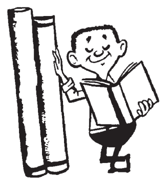
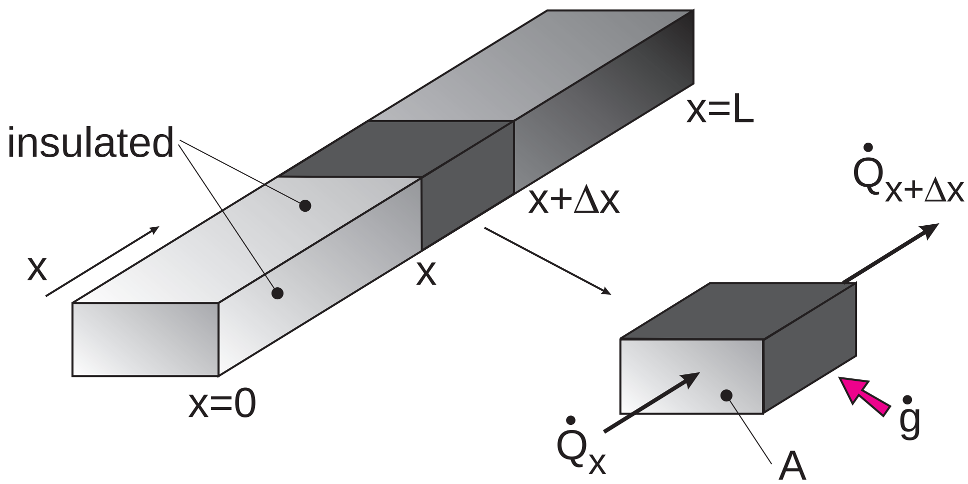
Conduction Heat Transfer



Reading	Problems
8-1 → 8-9	8-35, 8-63, 8-64, 8-69, 8-75, 8-80, 8-84, 8-89, 8-100, 8-103, 8-112, 8-126, 8-141, 8-150, 8-177
9-1 → 9-2	9-15, 9-20, 9-21, 9-36, 9-41, 9-42

Fourier Law of Heat Conduction

Consider a long solid bar where heat travels in a one-dimensional manner in the positive x direction as shown below.



We can examine a small representative element of the bar over the range $x \rightarrow x + \Delta x$. Just as in the case of a thermodynamic closed system, we can treat this slice as a *closed system* where no mass flows through any of the six faces.

From a 1^{st} law energy balance:

$$\frac{\partial E}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x} - \dot{W}$$

If the volume to the element is given as $V = A \cdot \Delta x$, then the mass of the element is

$$m = \rho \cdot A \cdot \Delta x$$

where ρ is the density of the material. The energy term ($KE = PE = 0$) is

$$E = m \cdot u = (\rho \cdot A \cdot \Delta x) \cdot u$$

For an incompressible substance the internal energy is $du = C\ dT$ and we can write

$$\frac{\partial E}{\partial t} = \rho C A \Delta x \frac{\partial T}{\partial t}$$

where we use the partial derivative because $T = T(x, t)$.

Given our earlier convention for $+ve$ work out of the body, we will assume that any electrical resistance heat addition to the body is denoted at $-\dot{W}$. Therefore

$$-\dot{W} = \dot{g} \cdot V = \dot{g} \cdot A \cdot \Delta x \quad \Leftarrow \quad \dot{g} = \text{volumetric heat addition, } (W/m^3)$$

We can also write the heat flow along the $x-$direction as a product of the temperature difference (the driving force for heat transfer) and a proportionality constant, based on experimental observation.

$$\dot{Q}_x = \frac{kA}{\Delta x}(T_x - T_{x+\Delta x})$$

where k is the thermal conductivity of the material. In the limit as $\Delta x \to 0$

$$\dot{Q}_x = -kA\frac{\partial T}{\partial x}$$

This is *Fourier's law of heat conduction.* The $-ve$ in front of k guarantees that we adhere to the 2^{nd} law and that heat always flows in the direction of lower temperature.

We can write the heat flow rate across the differential length, Δx as a truncated Taylor series expansion as follows

$$\dot{Q}_{x+\Delta x} = \dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x}\Delta x$$

when combined with Fourier's equation gives

$$\dot{Q}_{x+\Delta x} = \underbrace{-kA\frac{\partial T}{\partial x}}_{\dot{Q}_x} - \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)\Delta x$$

Noting that

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} = \frac{\partial E}{\partial t} + \dot{W} = \rho C A \Delta x \frac{\partial T}{\partial t} - A \Delta x \dot{g}$$

By removing the common factor of $A\Delta x$ we can then write the general 1-D conduction equation as

$$\underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)}_{\substack{\text{longitudinal}\\ \text{conduction}}} + \underbrace{\dot{g}}_{\substack{\text{internal}\\ \text{heat}\\ \text{generation}}} = \underbrace{\rho C \frac{\partial T}{\partial t}}_{\substack{\text{thermal}\\ \text{inertia}}}$$

Special Cases

1. Multidimensional Systems: The general conduction equation can be extended to three dimensional Cartesian systems as follows:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} = \rho C \frac{\partial T}{\partial t}$$

2. Constant Properties: If we assume that properties are independent of temperature, then the conductivity can be taken outside the derivative

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{g} = \rho C \frac{\partial T}{\partial t}$$

$$\nabla^2 T + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where

$$\nabla = \text{del operator} \equiv \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)$$

$$\alpha = \text{thermal diffusivity} \equiv \frac{k}{\rho C}$$

3. Steady State: If $t \to \infty$ then all terms $\dfrac{\partial}{\partial t} \to 0$

$$\nabla^2 T = -\frac{\dot{g}}{k} \qquad \Leftarrow \text{ Poisson's Equation}$$

4. No Internal Heat Generation:

$$\nabla^2 T = 0 \qquad \Leftarrow \text{ Laplace's Equation}$$

Thermal Resistance Networks

Resistances in Series

Conditions for 1-D heat flow through a plane wall include:

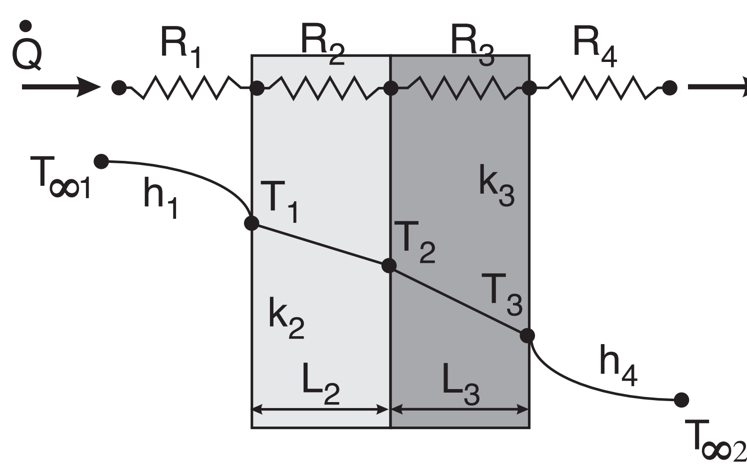
- constant cross sectional area, A
- steady flow conditions

The heat transfer across the fluid/solid interface is based on *Newton's law of cooling*

$$\dot{Q} = hA(T_{in} - T_{out}) = \frac{T_{in} - T_{out}}{R_{conv}} \qquad \text{where} \quad R_{conv} = \frac{1}{hA}$$

The heat flow through a solid material of conductivity, k is

$$\dot{Q} = \frac{kA}{L}(T_{in} - T_{out}) = \frac{T_{in} - T_{out}}{R_{cond}} \qquad \text{where} \quad R_{cond} = \frac{L}{kA}$$



By summing the temperature drop across each section, we can write:

$$\begin{aligned}
\dot{Q}\,R_1 &= (T_{\infty_1} - T_1) \\
\dot{Q}\,R_2 &= (T_1 - T_2) \\
\dot{Q}\,R_3 &= (T_2 - T_3) \\
\dot{Q}\,R_4 &= (T_3 - T_{\infty_2}) \\
\hline
\dot{Q}\left(\sum_{i=1}^{4} R_i\right) &= (T_{\infty_1} - T_{\infty_2})
\end{aligned}$$

The total heat flow across the system can be written as

$$\dot{Q} = \frac{T_{\infty_1} - T_{\infty_2}}{R_{total}} \qquad \text{where} \quad R_{total} = \sum_{i=1}^{4} R_i$$

This is analogous to current flow through electrical circuits where, $I = \Delta V/R$

Thermal	***Electrical***	***Role***
$\dot{Q}$, heat flow rate	I, current flow	transfer mechanism
$T_{in} - T_{out}$, temperature difference	$V_{in} - V_{out}$, voltage difference	driving force
R, thermal resistance	R, electrical resistance	impedance to flow

The heat flow rate is sometimes written in terms of an overall heat transfer coefficient, U

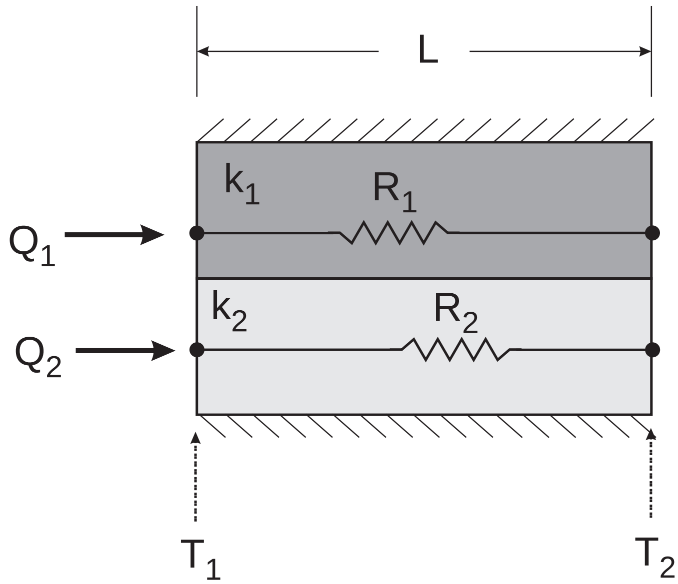
$$\dot{Q} = UA(T_{\infty_1} - T_{\infty_2})$$

where

$$UA = \frac{1}{R_{total}} = \frac{1}{\dfrac{1}{h_1 A} + \dfrac{L_2}{k_2 A} + \dfrac{L_3}{k_3 A} + \dfrac{1}{h_4 A}}$$

Resistances in Parallel

Consider the following system



For systems of parallel flow paths as shown above, we can use the $\mathbf{1}^{st}$ law to preserve the total energy

$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2$$

where we can write

$$\dot{Q}_1 = \frac{T_1 - T_2}{R_1} \qquad\qquad R_1 = \frac{L}{k_1 A_1}$$

$$\dot{Q}_2 = \frac{T_1 - T_2}{R_2} \qquad\qquad R_2 = \frac{L}{k_2 A_2}$$

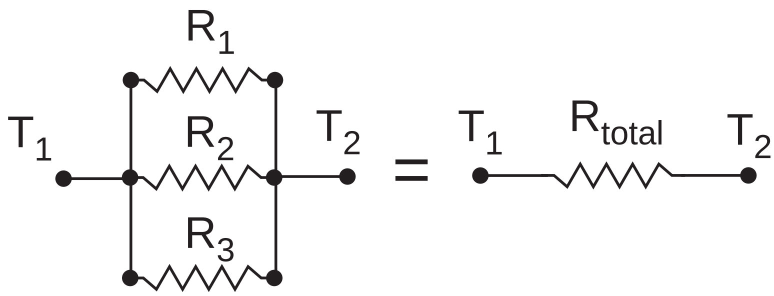
$$\dot{Q} = \sum \dot{Q}_i = (T_1 - T_2)\left(\sum \frac{1}{R_i}\right) \qquad \text{where} \qquad \frac{1}{R_{total}} = \sum \frac{1}{R_i}$$

Will temperature be a function of x only?

Or will heat flow across the interior interface?

} for these conditions, $T = T(x)$

In general, for parallel networks we can use a parallel resistor network as follows:



where

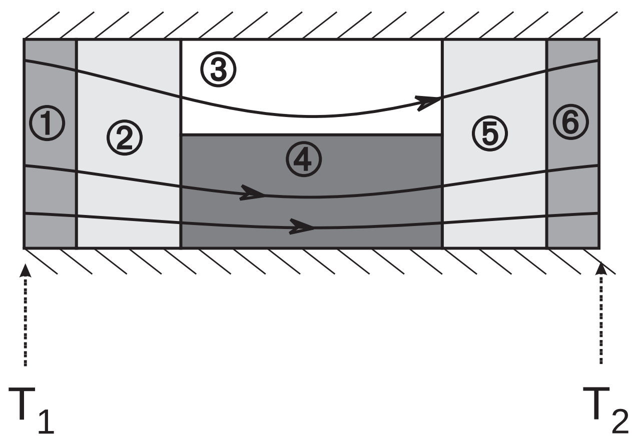
$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

and

$$\dot{Q} = \frac{T_1 - T_2}{R_{total}}$$

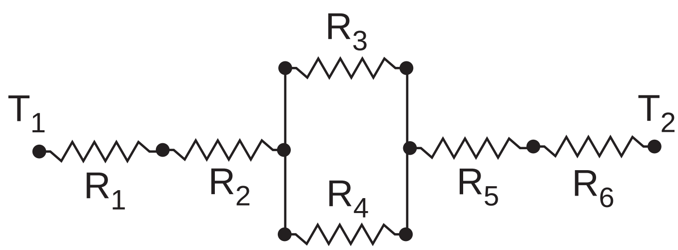
Flow in a Composite Wall

For a composite system where $\boldsymbol{k_3} << \boldsymbol{k_4}$ the heat flow lines would look like



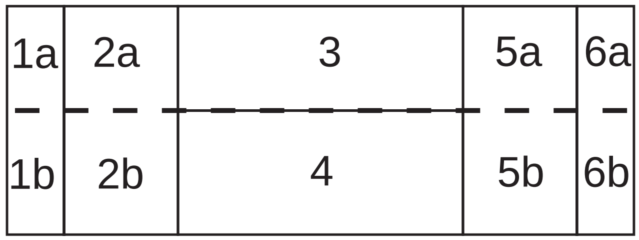
This problem cannot be easily solved using only a series network or a parallel network *but* we can use these two approaches as bounds on the problem and hopefully find an average of the two limits that will provide a reasonable compromise.

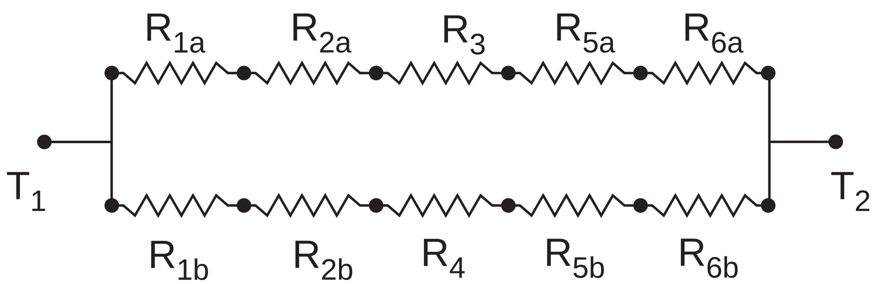
a) Upper bound on heat transfer: If we assume that heat flows in a 1-D manner within each section we can use a combined series/parallel resistor network as follows:



Since there is no turning in the heat flow path, the thermal resistance will be minimized and we will have an upper bound on the heat transfer.

b) Lower bound on heat transfer: The lower bound on heat transfer (upper bound on resistance) is achieved by forcing heat flow through an upper and a lower path as shown below.

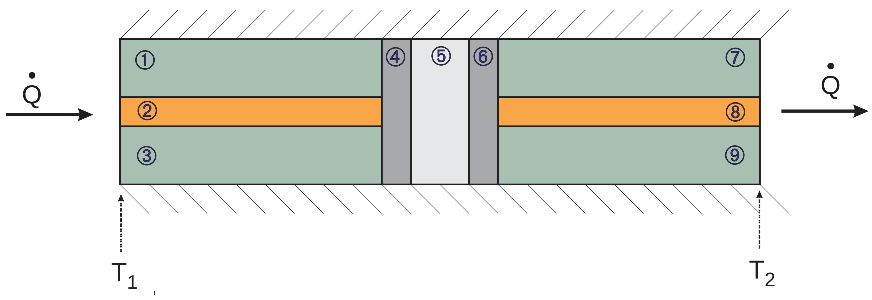




- equivalent to inserting a thin insulator along the dotted line
  - forcing heat flow through the upper and lower paths
  - prevents heat flow from finding a path of least resistance
  - leads to a lower bound on heat transfer
- since the solid resistance in each element is $\boldsymbol{R = \frac{L}{kA}}$ the area used in the upper and lower sections will be adjusted to reflect the cross sectional flow area of that section

Example: Resistance in a Composite Wall

Consider heat flow rate through a section of a printed circuit board as shown below. Use the upper and lower bound on heat flow to find an effective heat flow rate and an effective thermal resistance.



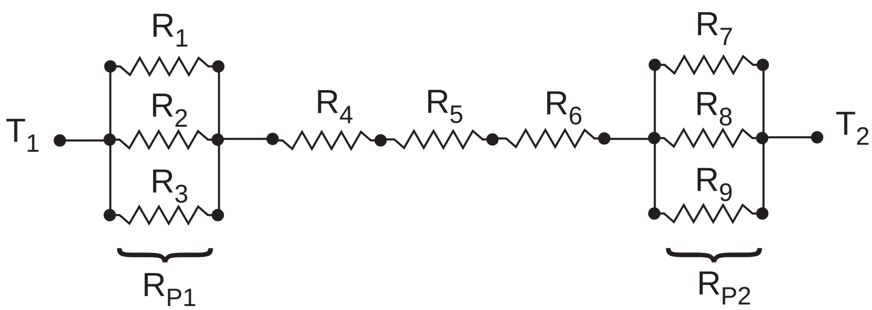
section	material	conductivity W/mK	thickness mm	length mm
1	FR4	0.4	1.0	10
2	Cu	400	0.1	10
3	FR4	0.4	1.0	10
4	Cu	400	2.1	0.1
5	solder	36	2.1	1.0
6	Cu	400	2.1	0.1
7	FR4	0.4	1.0	10
8	Cu	400	0.1	10
9	FR4	0.4	1.0	10

Assume the width of the test section to be $W = 10\ mm$.

Temperatures:

$T_1 = 100\ ^\circ C$ $\qquad\qquad$ $T_2 = 50\ ^\circ C$

Upper bound on heat transfer



$$
\begin{aligned}
R_1 = R_3 &= \frac{L_1}{k_1 A_1} = \frac{0.01\ m}{0.4\ W/mK \cdot 0.001\ m \cdot 0.01\ m} = 2500\ K/W \\
R_2 &= \frac{L_2}{k_2 A_2} = \frac{0.01\ m}{400\ W/mK \cdot 0.0001\ m \cdot 0.01\ m} = 25\ K/W \\
R_4 = R_6 &= \frac{L_4}{k_4 A_4} = \frac{0.0001\ m}{400\ W/mK \cdot 0.0021\ m \cdot 0.01\ m} = 0.012\ K/W \\
R_5 &= \frac{L_5}{k_5 A_5} = \frac{0.001\ m}{36\ W/mK \cdot 0.0021\ m \cdot 0.01\ m} = 1.323\ K/W \\
R_7 = R_9 &= \frac{L_7}{k_7 A_7} = \frac{0.01\ m}{0.4\ W/mK \cdot 0.001\ m \cdot 0.01\ m} = 2500\ K/W \\
R_8 &= \frac{L_8}{k_8 A_8} = \frac{0.01\ m}{400\ W/mK \cdot 0.0001\ m \cdot 0.01\ m} = 25\ K/W \\
\frac{1}{R_{p1}} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{2500\ K/W} + \frac{1}{25\ K/W} + \frac{1}{2500\ K/W} \\
R_{p1} = R_{p2} &= 24.510\ K/W
\end{aligned}
$$

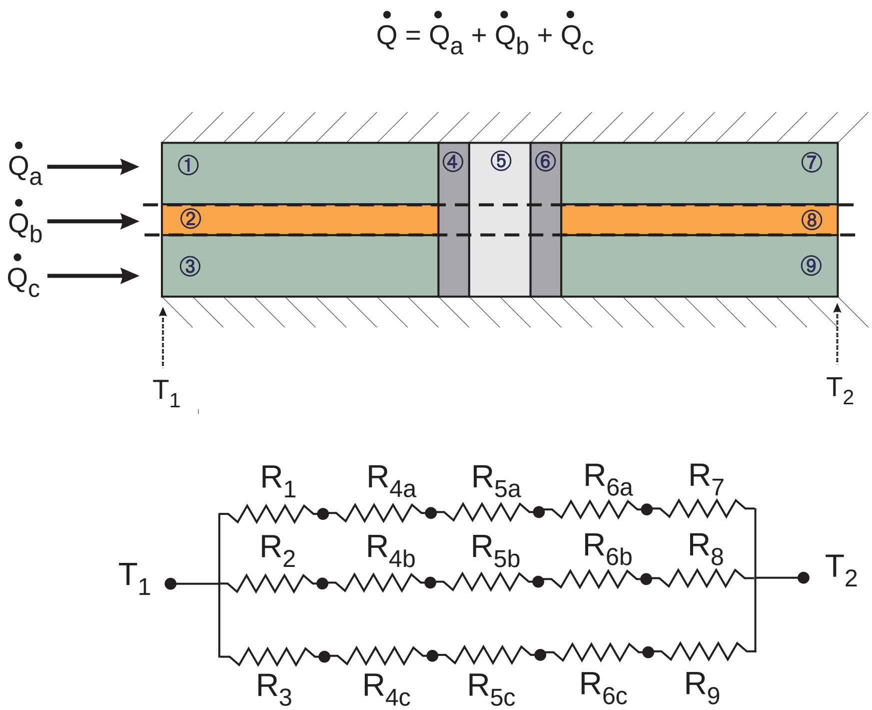
The lower bound for resistance is given as

$$
\begin{aligned}
R_{total} &= R_{p1} + R_4 + R_5 + R_6 + R_{p2} \\
&= (24.510 + 0.012 + 1.323 + 0.012 + 24.510) = 50.367\ K/W
\end{aligned}
$$

and the upper bound for heat transfer is

$$\dot{Q} = \frac{(T_1 - T_2)}{R_{total}} = \frac{(100 - 50)\ K}{50.367\ K/W} = 0.993\ W \Leftarrow$$

Lower bound on heat transfer



As shown in the resistance network, the lower on heat transfer is obtained by restricting the heat flow to three predefined paths.

$$R_1 = R_3 = \frac{L_1}{k_1 A_1} = \frac{0.01\ m}{0.4\ W/mK \cdot 0.001\ m \cdot 0.01\ m} = 2500\ K/W$$

$$R_2 = \frac{L_2}{k_2 A_2} = \frac{0.01\ m}{400\ W/mK \cdot 0.0001\ m \cdot 0.01\ m} = 25\ K/W$$

$$R_{4a} = R_{6a} = \frac{L_4}{k_4 A_{4a}} = \frac{0.0001\ m}{400\ W/mK \cdot 0.001\ m \cdot 0.01\ m} = 0.025\ K/W$$

$$
\begin{aligned}
R_{4b} = R_{6b} &= \frac{L_4}{k_4 A_{4b}} = \frac{0.0001\, m}{400\, W/mK \cdot 0.0001\, m \cdot 0.01\, m} = 0.25\, K/W \\
R_{4c} = R_{6c} &= \frac{L_4}{k_4 A_{4c}} = \frac{0.0001\, m}{400\, W/mK \cdot 0.001\, m \cdot 0.01\, m} = 0.025\, K/W \\
R_{5a} &= \frac{L_5}{k_5 A_{5a}} = \frac{0.001\, m}{36\, W/mK \cdot 0.001\, m \cdot 0.01\, m} = 2.778\, K/W \\
R_{5b} &= \frac{L_5}{k_5 A_{5b}} = \frac{0.001\, m}{36\, W/mK \cdot 0.0001\, m \cdot 0.01\, m} = 27.778\, K/W \\
R_{5c} &= \frac{L_5}{k_5 A_{5c}} = \frac{0.001\, mm}{36\, W/mK \cdot 0.001\, m \cdot 0.01\, m} = 2.778\, K/W \\
R_7 = R_9 &= \frac{L_7}{k_7 A_7} = \frac{0.01\, m}{0.4\, W/mK \cdot 0.001\, m \cdot 0.01\, m} = 2500\, K/W \\
R_8 &= \frac{L_8}{k_8 A_8} = \frac{0.01\, m}{400\, W/mK \cdot 0.0001\, m \cdot 0.01\, m} = 25\, K/W \\
R_a &= R_1 + R_{4a} + R_{5a} + R_{6a} + R_7 \\
&= 2500 + 0.025 + 2.778 + 0.025 + 2500 = 5002.828\, K/W \\
R_b &= R_2 + R_{4b} + R_{5b} + R_{6b} + R_8 \\
&= 25 + 0.25 + 27.778 + 0.25 + 25 = 78.278\, K/W \\
R_c &= R_3 + R_{4c} + R_{5c} + R_{6c} + R_9 \\
&= 2500 + 0.025 + 2.778 + 0.025 + 2500 = 5002.828\, K/W \\
\frac{1}{R_{total}} &= \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} = \frac{1}{5002.828} + \frac{1}{78.278} + \frac{1}{5002.828} \\
R_{total} &= 75.90\, K/W
\end{aligned}
$$

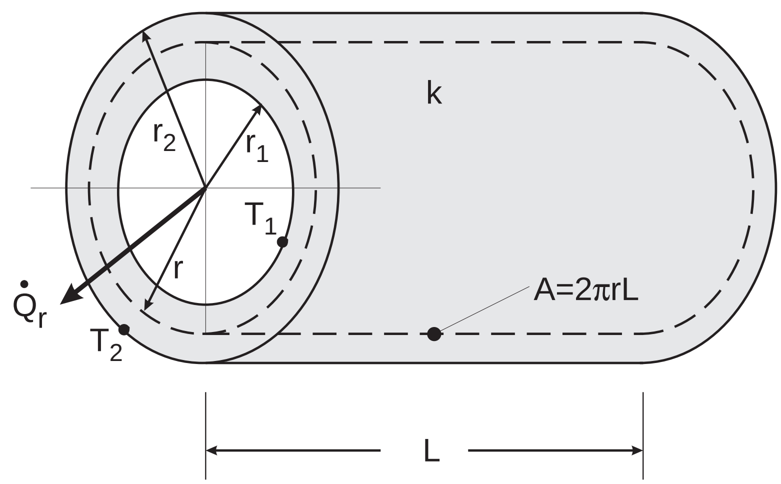
We can then use this value of resistance to find the lower bound for heat transfer

$$
\dot{Q} = \frac{(T_1 - T_2)}{R_{total}} = \frac{(100 - 50)\, K}{75.90\, K/W} = 0.659\, W \Leftarrow
$$

In summary

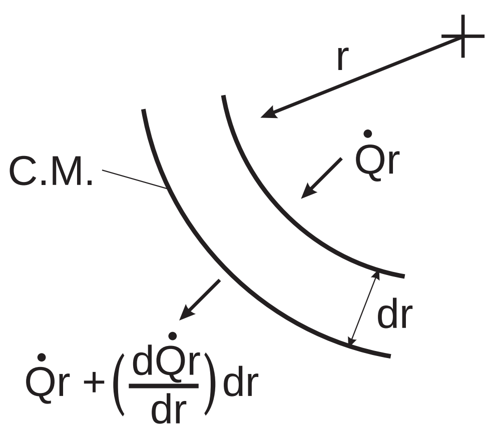
	Resistance (K/W)		Heat Transfer (W)
lower bound	50.36	$\searrow$	0.659
upper bound	75.90	$\nearrow$	0.993

Cylindrical Systems



Steady, 1D heat flow from T_1 to T_2 in a cylindrical systems occurs in a radial direction where the lines of constant temperature (isotherms) are concentric circles, as shown by the dotted line in the figure above and $T = T(r)$.

Performing a 1^{st} law energy balance on a *control mass* from the annular ring of the cylindrical cylinder



$$\frac{dE_{C.M.}}{dt}^{\nearrow 0} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen}^{\nearrow 0}$$

Therefore

$$\dot{Q}_r - \left[\dot{Q}_r + \left(\frac{d\dot{Q}_r}{dr}\right) dr\right] = 0 \qquad \Rightarrow \qquad \frac{d\dot{Q}_r}{dr} = 0$$

Integrating this result we obtain

$$\dot{Q}_r = constant$$

Using Fourier's law of heat conduction

$$\dot{Q}_r = -k\, A\, \frac{dT}{dr} = constant$$

where the cross sectional area for heat flow is given as the product of the circumference of the circular ring and the length of the cylinder

$$A = 2\pi r \mathcal{L}$$

Separating the above equation gives

$$dT = -\frac{\dot{Q}}{2\pi k \mathcal{L}}\, \frac{dr}{r}$$

Integrating from $T = T_1$ @ $r = r_1$ to $T = T_2$ @ $r = r_2$

$$T_2 - T_1 = -\frac{\dot{Q}_r}{2\pi k \mathcal{L}}(\ln r_2 - \ln r_1) = -\frac{\dot{Q}_r}{2\pi k \mathcal{L}} \ln \frac{r_2}{r_1}$$

Therefore we can write

$$\dot{Q}_r = \frac{T_1 - T_2}{\left(\dfrac{\ln(r_2/r_1)}{2\pi k \mathcal{L}}\right)} \qquad \text{where} \quad R = \left(\frac{\ln(r_2/r_1)}{2\pi k \mathcal{L}}\right)$$

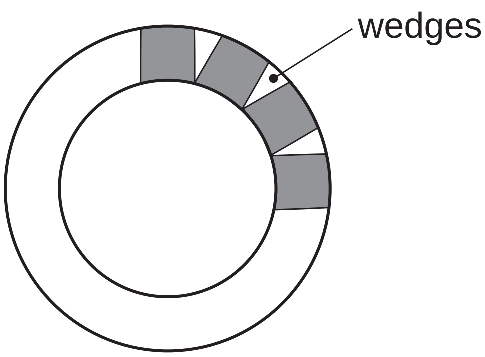
Note: If we let the flow length be $L = r_2 - r_1$, the resistance can be written as

$$R = \frac{\ln(1 + L/r_1)}{2\pi k \mathcal{L}}$$

In the limit as $L/r_1 \rightarrow 0$ (i.e. for a thin annulus or a large inner radius)

$$R = \frac{L/r_i}{2\pi k \mathcal{L}} = \frac{L}{\underbrace{2\pi r_1 \mathcal{L}}_{A_1}\ k} = \frac{L}{k A_1}$$

The cylinder and the slab resistance become the same as the wall thickness becomes small compared to the inner radius.

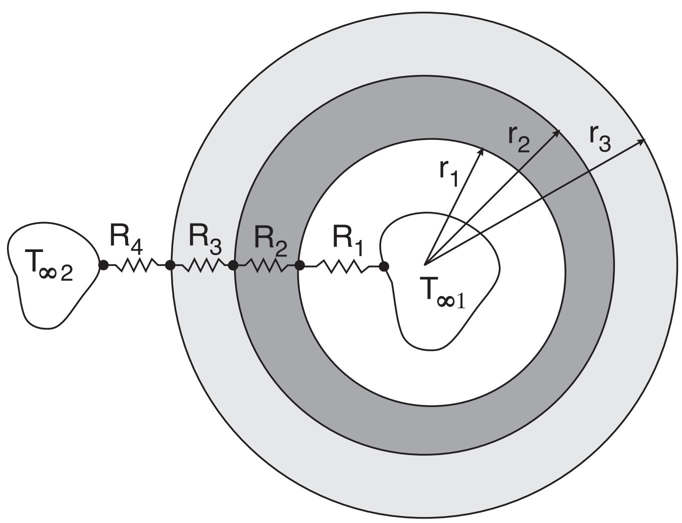


The slab resistance is equivalent to the cylinder resistance if the wedge shaped material is removed.

Composite Cylinders

We know from the slab that

$$\dot{Q} = \frac{T_{\infty_1} - T_{\infty_2}}{R_{total}} \qquad \text{where} \quad R_{total} = \sum_{i=1}^{n} R_i$$



If the outer and inner boundary layer convective resistances are written as

$$R_1 = \frac{1}{h_1 A_1} \qquad \text{and} \qquad R_4 = \frac{1}{h_4 A_4}$$

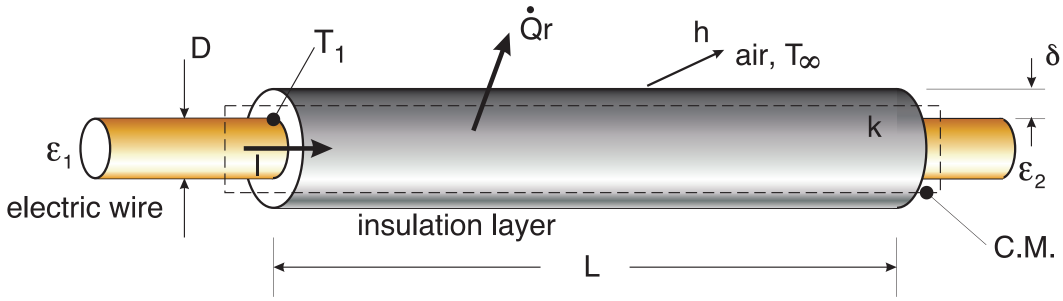
and the solid resistance in the two annular regions

$$\boldsymbol{R_2 = \frac{\ln(r_2/r_1)}{2\pi k_2 \mathcal{L}}} \qquad \text{and} \qquad \boldsymbol{R_3 = \frac{\ln(r_3/r_2)}{2\pi k_3 \mathcal{L}}}$$

Then the total resistance can be written as

$$\begin{aligned} \boldsymbol{R_{total}} &= \boldsymbol{R_1 + R_2 + R_3 + R_4} \\ &= \boldsymbol{\frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi k_2 \mathcal{L}} + \frac{\ln(r_3/r_2)}{2\pi k_3 \mathcal{L}} + \frac{1}{h_4 A_4}} \end{aligned}$$

Example: Heat Loss from an Electric Wire



Given:

$$I = 10\ A$$

$$\begin{aligned}
\Delta\epsilon = \epsilon_1 - \epsilon_2 &= 8\ V \\
D &= 3\ mm \\
\mathcal{L} &= 5\ m \\
k &= 0.15\ W/mK \\
T_\infty &= 30\ ^\circ C \\
h &= 12\ W/m^2 \cdot K
\end{aligned}$$

Find:

$$\begin{aligned}
T_1 \text{ when } \delta &= 2\ mm, \text{ and} \\
\delta &= 4\ mm
\end{aligned}$$

Solution:

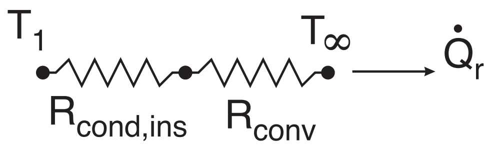
From the 1^{st} law for a C.M. system

$$\frac{dE_{C.M.}}{dt}\nearrow^0 = \underbrace{\dot{W}_e}_{I\Delta\epsilon} - \dot{Q}_r$$

Therefore

$$\dot{Q}_r = I\Delta\epsilon = 10A \times 8\ V = 80\ W$$

The thermal resistance network gives



$$R_{cond,ins} = \frac{\ln(r_2/r_1)}{2\pi k \mathcal{L}}$$

$$R_{conv} = \frac{1}{hA_2}$$

where

$$r_1 = D/2$$

$$r_2 = \frac{D}{2} + \delta$$

$$A_2 = 2\pi r_2 \mathcal{L}$$

Case 1: $r_1 = 1.5\ mm$ and $r_2 = 3.5\ mm$

$$R_{cond,ins} = \frac{\ln(3.5/1.5)}{2\pi \times 0.15\ W/mK \times 5\ m} = 0.1798\ K/W$$

$$R_{conv} = \frac{1}{12\ W/m^2 \cdot K \times 2\pi \times 3.5 \times 10^{-3}\ m \times 5\ m} = 0.7579\ K/W$$

$$R_{total} = R_{conv} + R_{cond,ins} = 0.9377\ K/W$$

and

$$\dot{Q}_r = \frac{T_1 - T_\infty}{R_{total}} \quad \Rightarrow \quad T_1 = T_\infty + \dot{Q}_r R_{total} \approx 105\ ^\circ C$$

Case 2: $r_1 = 1.5\ mm$ and $r_2 = 5.5\ mm$

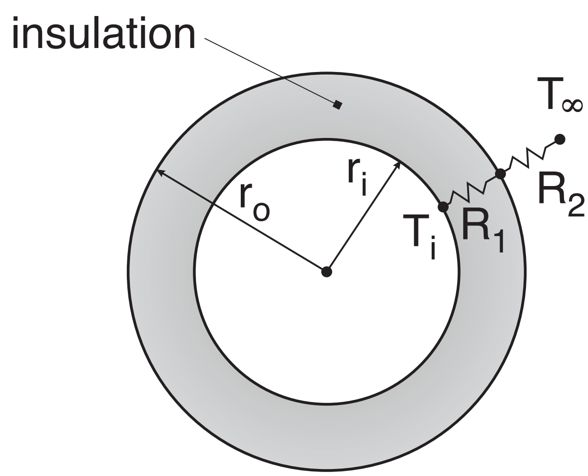
$$\begin{aligned} R_{cond,ins} &= \frac{\ln(5.5/1.5)}{2\pi \times 0.15\ W/mK \times 5\ m} = 0.2757\ K/W \\ R_{conv} &= \frac{1}{12\ W/m^2 \cdot K \times 2\pi \times 5.5 \times 10^{-3}\ m \times 5\ m} = 0.4823\ K/W \\ R_{total} &= R_{conv} + R_{cond,ins} = 0.7580\ K/W \end{aligned}$$

and

$$\dot{Q}_r = \frac{T_1 - T_\infty}{R_{total}} \quad \Rightarrow \quad T_1 = T_\infty + \dot{Q}_r R_{total} \approx 90.64\ ^\circ C$$

Why does the temperature of the wire T_1 decrease when the thickness of the insulation layer doubles?

Critical Thickness of Insulation



Consider a steady, 1-D problem where an insulation cladding is added to the outside of a tube with constant surface temperature T_i. What happens to the heat transfer as insulation is added, i.e. we increase the thickness of the insulation?

The resistor network can be written as a series combination of the resistance of the insulation, R_1 and the convective resistance, R_2

$$R_{total} = R_1 + R_2 = \frac{\ln(r_o/r_i)}{2\pi k \mathcal{L}} + \frac{1}{h 2\pi r_o \mathcal{L}}$$

Note: as the thickness of the insulation is increased the outer radius, r_o increases.

Although the purpose of adding more insulation is to increase resistance and decrease heat transfer, we can see from the resistor network that increasing r_o actually results in a decrease in the convective resistance.

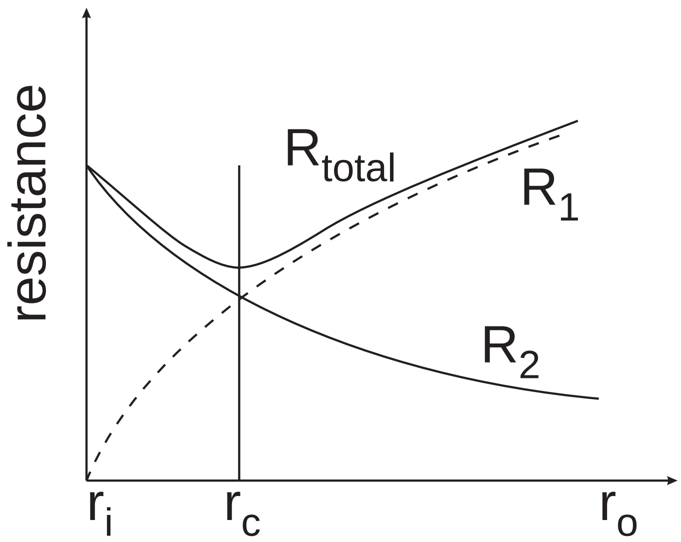
Could there be a situation in which adding insulation <u>increases</u> the overall heat transfer?

To find the critical radius, r_c, where adding more insulation begins to decrease heat transfer, set

$$\frac{dR_{total}}{dr_o} = 0$$

$$\frac{dR_{total}}{dr_o} = \frac{1}{2\pi k r_o \mathcal{L}} - \frac{1}{h 2\pi r_o^2 \mathcal{L}} = 0$$

$$r_c = \frac{k}{h}$$

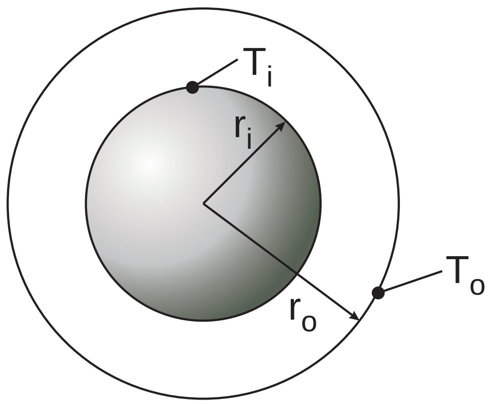


There is always a value of r_c, but there is a minimum in heat transfer only if $r_c > r_i$

As an example for natural convection + radiation $h \approx 8.5$

Glass wool:	$k = 0.038\ [W/(m \cdot K)]$	$r_c = 4.5\ mm$
Rubber:	$k = 0.16\ [W/(m \cdot K)]$	$r_c = 18\ mm$
Copper:	$k = 401\ [W/(m \cdot K)]$	$r_c = 47\ mm \leftarrow$ fin

Spherical Systems



For steady, 1D heat flow in spherical geometries we can use a 1^{st} law energy balance similar to the analysis for cylindrical systems to obtain

$$\dot{Q} = \left(-k\frac{dT}{dr}\right)\underbrace{4\pi r^2}_{A} = \text{constant}$$

Then integrating from the inner to the outer radius

$$\int_{T_i}^{T_o} dT = -\int_{r_i}^{r_o} \frac{\dot{Q}}{4\pi r^2 k}\, dr$$

$$T_o - T_i = \frac{\dot{Q}}{4\pi k}\left[\frac{1}{r_o} - \frac{1}{r_i}\right]$$

We can write the heat transfer as

$$\dot{Q} = \frac{4\pi k r_i r_o}{(r_0 - r_i)}\,(T_i - T_o) = \frac{(T_i - T_o)}{R} \qquad \text{where} \quad R = \frac{r_o - r_i}{4\pi k r_i r_o}$$

Notes:

1. as $r_0 \rightarrow r_i, \qquad R \rightarrow \frac{L}{kA}; \qquad L = r_0 - r_i$

Heat Generation in a Solid

Heat can be generated within a solid as a result of:

- resistance heating in wires
- chemical reactions
- nuclear reactions

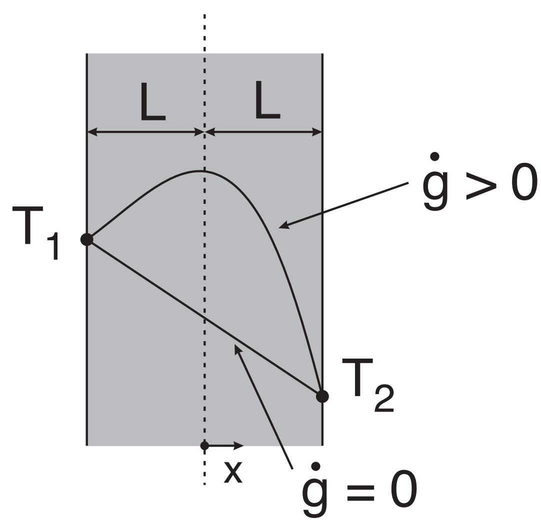
A volumetric heat generation terms will be defined as follows:

$$\dot{g} = \frac{\dot{E}_g}{V} \quad (W/m^3)$$

for heat generation in wires, we will define $\dot{g}$ as

$$\dot{g} = \frac{I^2 R_e}{\pi r_o^2 \mathcal{L}}$$

Slab System



Assume steady, 1D conduction with constant $\dot{g}$ in the slab and constant k.

We can write the 1D conduction equation for a steady system as

$$k\frac{d^2T}{dx^2} + \dot{g} = 0$$

with boundary conditions

$$T = T_1 \quad @ \quad x = -L$$

$$T = T_2 \quad @ \quad x = +L$$

Integrating twice gives

$$T = -\frac{\dot{g}x^2}{2k} + C_1 x + C_2$$

From the boundary conditions we can find

$$\begin{aligned} C_2 &= \frac{(T_1 + T_2)}{2} + \frac{\dot{g}L^2}{2k} \\ C_1 &= -\left(\frac{T_1 - T_2}{2L}\right) \end{aligned}$$

Therefore

$$T = \frac{T_1 + T_2}{2} - \left(\frac{T_1 - T_2}{2}\right)\frac{x}{L} + \frac{\dot{q}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)$$

Case 1: if $T_1 = T_2 = T_s$

$$T = T_s + \frac{\dot{g}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)$$

This is a concave-down parabola. Why must the shape be of this form?

Case 2: If there is also heat loss from the outside edges to T_∞

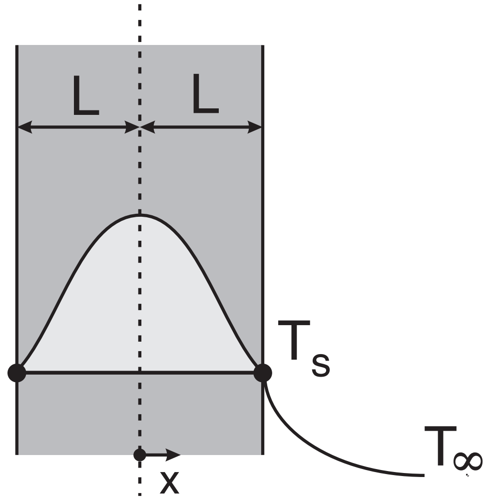
$$\dot{q} = -\left[k\frac{\partial T}{\partial x}\right]_{x=L} = -k\left\{\frac{\dot{g}L^2}{2k}\left(-\frac{2x}{L^2}\right)\right\}_{x=L} = +\dot{g}L$$

but Newton's law of cooling tells us

$$\dot{q} = h(T_s - T_\infty)$$

Therefore

$$T_s = T_\infty + \frac{\dot{g}L}{h}$$



Cylindrical System

The general conduction equation for a cylindrical system is given as

$$\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right)+\dot{g}=0$$

Integrating once gives

$$-2\pi r\mathcal{L}k\frac{dT}{dr}=(\pi r^2\mathcal{L})\dot{g}$$

While integrating a second time and noting the boundary condition $T=T_s$ @$r=r_0$, gives

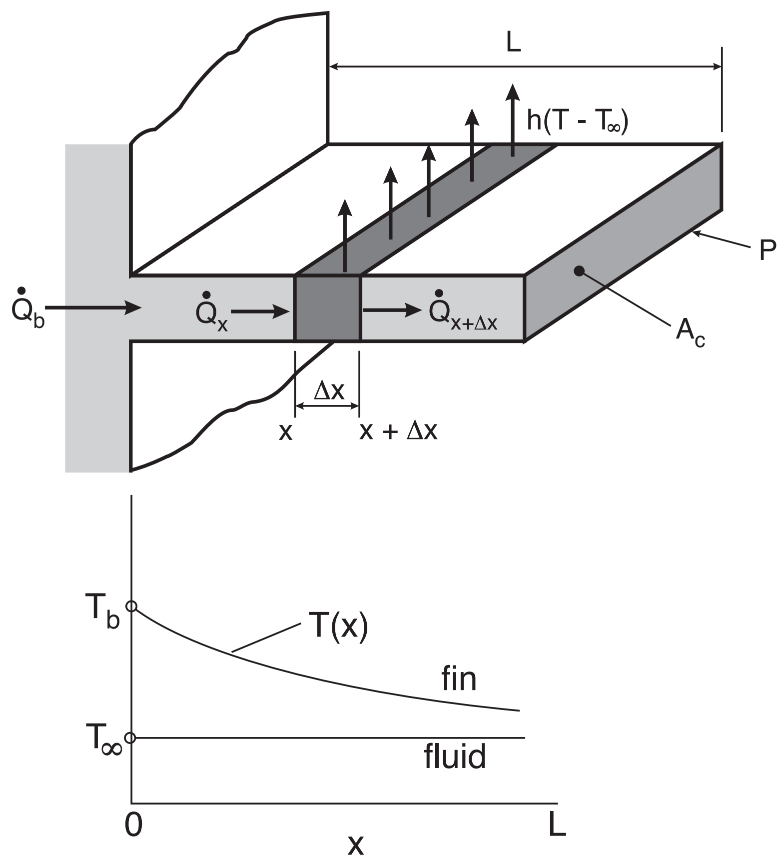
$$T=T_s+\frac{\dot{g}r_0^2}{4k}\left(1-\left(\frac{r}{r_0}\right)^2\right)$$

Heat Transfer from Finned Surfaces

The rate of heat transfer from a surface is given by *Newton's Law of Cooling*

$$\dot{Q} = hA(T_s - T_\infty)$$

- heat transfer is improved by $h \uparrow$ and $A \uparrow$
- it is difficult to obtain a significant increase in h
- an increase in A is easily achieved through extended surfaces or fins



We will assume a constant cross sectional area and that the temperature inside the fin is *only* a function of x.

Just as in the development for Fourier's law of conduction, we can establish a 1^{st} law balance over the thin slice of the fin between x and $x + \Delta x$ such that

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} - P\Delta x h(T - T_\infty) = 0$$

From Fourier's law we know

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} = kA_c \frac{d^2T}{dx^2} \Delta x$$

Therefore the conduction equation for a fin with constant cross section is

$$\underbrace{kA_c \frac{d^2T}{\partial x^2}}_{\substack{\text{longitudinal} \\ \text{conduction}}} - \underbrace{hP(T - T_\infty)}_{\substack{\text{lateral} \\ \text{convection}}} = 0$$

The temperature difference between the fin and the surroundings (temperature excess) is usually expressed as

$$\theta = T(x) - T_\infty$$

which allows the 1-D fin equation to be written as

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

where the fin parameter m is

$$m = \left(\frac{hP}{kA_c}\right)^{1/2}$$

and the boundary conditions are

$$\theta = \theta_b \quad @ \quad x = 0$$

$$\theta \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty$$

The solution to the differential equation for θ is

$$\theta(x) = C_1 \sinh(mx) + C_2 \cosh(mx)$$

substituting the boundary conditions to find the constants of integration

$$\theta = \theta_b \, \frac{\cosh[m(L-x)]}{\cosh(mL)}$$

The heat transfer flowing through the base of the fin can be determined as

$$\begin{aligned}\dot{Q}_b &= A_c \left(-k \frac{dT}{dx}\right)_{@x=0} \\ &= \theta_b (k A_c h P)^{1/2} \; \tanh(mL)\end{aligned}$$

Fin Efficiency and Effectiveness

The dimensionless parameter that compares the actual heat transfer from the fin to the ideal heat transfer from the fin is the *fin efficiency*

$$\eta = \frac{\text{actual heat transfer rate}}{\begin{array}{c}\text{maximum heat transfer rate when} \\ \text{the entire fin is at } T_b\end{array}} = \frac{\dot{Q}_b}{hPL\theta_b}$$

If the fin has a constant cross section then

$$\eta = \frac{\tanh(mL)}{mL}$$

An alternative figure of merit is the *fin effectiveness* given as

$$\epsilon_{fin} = \frac{\text{total fin heat transfer}}{\begin{array}{c}\text{the heat transfer that would have} \\ \text{occurred through the base area} \\ \text{in the absence of the fin}\end{array}} = \frac{\dot{Q}_b}{hA_c\theta_b}$$

Transient Heat Conduction

Various applications of transient heat conduction include:

- surface hardness control
- stress relieving
- processes requiring chemical reaction (i.e. curing of rubber)
- wire drawing and coating
- welding

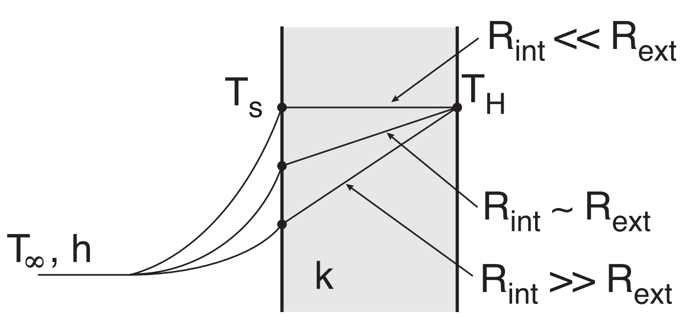
Performing a $\mathbf{1^{st}}$ law energy balance on a plane wall gives

$$\dot{E}_{in} - \dot{E}_{out} \;\Rightarrow \dot{Q}_{cond} = \frac{T_H - T_s}{L/(k \cdot A)} = \dot{Q}_{conv} = \frac{T_s - T_\infty}{1/(h \cdot A)}$$

where

$$\frac{T_H - T_s}{T_s - T_\infty} = \frac{L/(k \cdot A)}{1/(h \cdot A)} = \frac{\text{internal resistance to H.T.}}{\text{external resistance to H.T.}}$$

$$= \frac{hL}{k} = Bi \equiv \text{Biot number}$$



$R_{int} << R_{ext}$**:** the Biot number is small and we can conclude

$$T_H - T_s << T_s - T_\infty \qquad \text{and in the limit} \quad T_H \approx T_s$$

$R_{ext} << R_{int}$:

$R_{int} << R_{ext}$: the Biot number is large and we can conclude

$$T_s - T_\infty << T_H - T_s \qquad \text{and in the limit} \quad T_s \approx T_\infty$$

Lumped System Analysis

- if the internal temperature of a body remains relatively constant with respect to time
  - can be treated as a lumped system analysis
  - heat transfer is a function of time only, $T = T(t)$
- internal temperature is relatively constant at low Biot number
- typical criteria for lumped system analysis $\longrightarrow$ $Bi \leq 0.1$

Example 1: Heat transfer from an extended surface

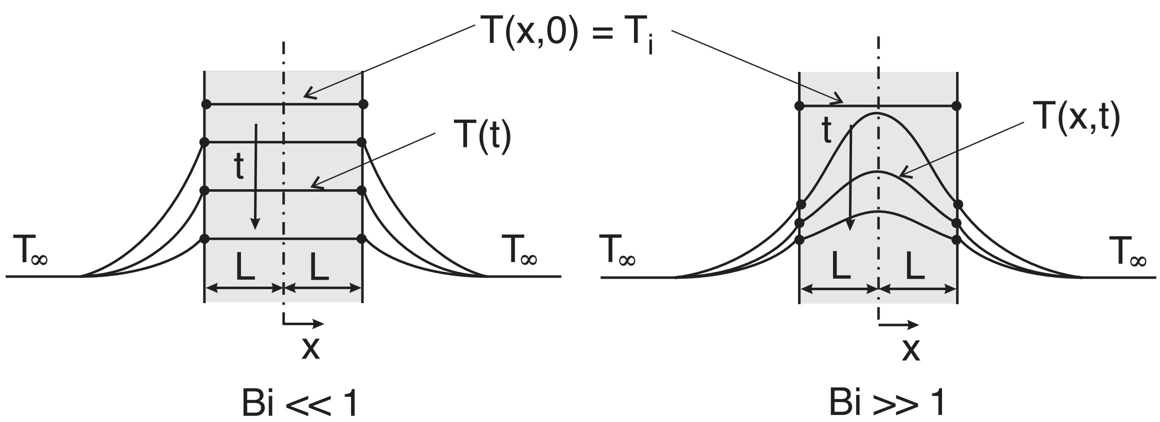
Material:	Aluminum, $k = 200\ W/mK$
Fin Thickness:	$2\ mm$
Heat transfer coefficient:	forced convection, $h = 25\ W/m^2K$

The Biot number is given as

$$Bi = \frac{h \cdot \delta/2}{k} = \frac{25\ W/m^2K \cdot (0.002/2)\ m}{200\ W/mK} = 0.000125\ (<< 0.1)$$

We can clearly use a lumped system analysis in this instance.

Example 2: A plane wall uniformly at temperature T_i is suddenly exposed to a cooler ambient at T_∞

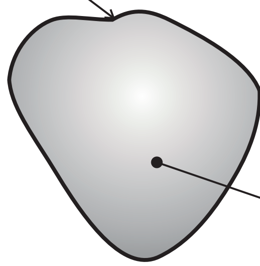


The temperature distribution within the wall remains spatially uniform but temporally varying. That is

$$T = T(t)$$

To a very good approximation, this spatial uniformity form the basis for the lumped analysis.

Transient Conduction Analysis

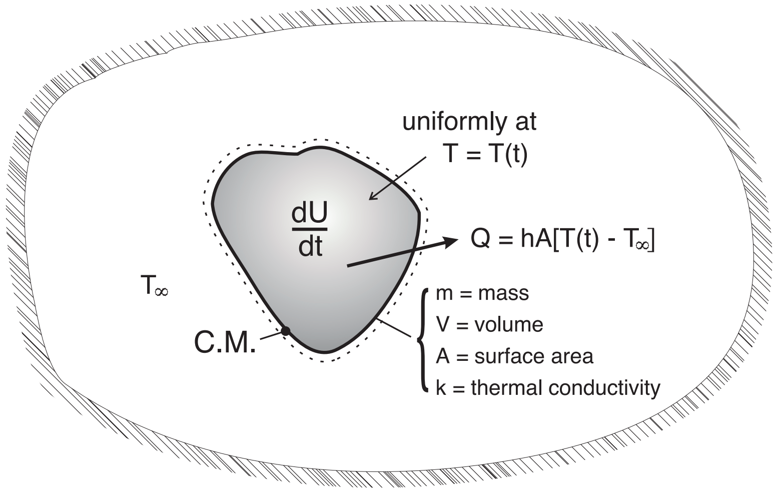


For the 3-D body of volume V and surface area A, we can use a lumped system analysis if

$$Bi = \frac{hV}{kA} < 0.1 \qquad \Leftarrow \text{results in an error of less that } 5\%$$

The characteristic length for the 3-D object is given as $\mathcal{L} = V/A$. Other characteristic lengths for conventional bodies include:

Slab	$\frac{V}{A_s} = \frac{WH2L}{2WH} = L$
Rod	$\frac{V}{A_s} = \frac{\pi r_o^2 L}{2\pi r_0 L} = \frac{r_0}{2}$
Sphere	$\frac{V}{A_s} = \frac{4/3\pi r_o^3}{4\pi r_0^2} = \frac{r_0}{3}$



At $t > 0, \;\; T = T(x, y, z, t)$, however, when $Bi < 0.1$ then we can assume $T \approx T(t)$.

Performing a 1^{st} law energy balance on the control volume shown below

$$\frac{dE_{C.M.}}{dt} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_g \nearrow^0$$

If we assume PE and KE to be negligible then

$$\frac{dU}{dt} = -\dot{Q} \qquad \Leftarrow \frac{dU}{dt} < 0 \text{ implies } U \text{ is decreasing}$$

For an incompressible substance specific heat is constant and we can write

$$\underbrace{mC}_{\equiv C_{th}} \frac{dT}{dt} = - \underbrace{Ah}_{1/R_{th}} (T - T_\infty)$$

where

$$C_{th} = \text{lumped capacitance}$$

It should be clearly noted that we have neglected the spatial dependence of the temperature within the object. This type of an approach is only valid for $Bi = \dfrac{hV}{kA} < 0.1$

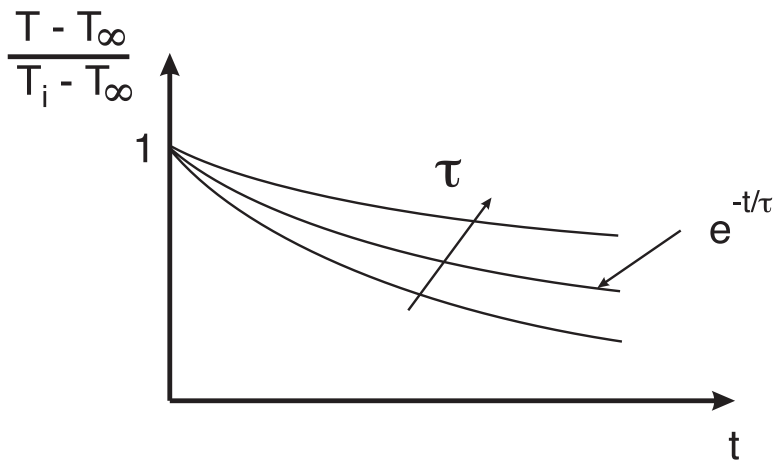
$$C_{th} \frac{dT}{dt} = -\frac{1}{R_{th}} (T - T_\infty)$$

We can integrate and apply the initial condition, $T = T_i \ @t = 0$ to obtain

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-t/(R_{th} \cdot C_{th})} = e^{-t/\tau}$$

where

$$\tau = R_{th} \cdot C_{th} = \text{thermal time constant} = \frac{mC}{Ah}$$



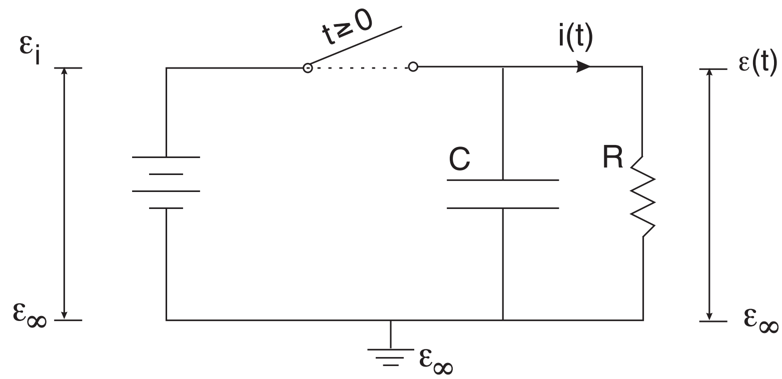
The total heat transfer rate can be determined by integrating $\dot{Q}$ with respect to time.

$$\begin{aligned}
\dot{Q}_{total} &= \int_0^{t^*} \dot{Q}\, dt = \int_0^{t^*} hA(T - T_\infty)\, dt \\
&= hA(T_i - T_\infty) \int_0^{t^*} e^{-t/\tau}\, dt \\
&= hA(T_i - T_\infty)(\tau)[1 - e^{-t^*/\tau}]
\end{aligned}$$

Therefore

$$\dot{Q}_{total} = \dot{m}C(T_i - T_\infty)[1 - e^{-t^*/\tau}]$$

Electrical Analogy



$$I(t) = \frac{\epsilon(t) - \epsilon_\infty}{R} = -C\left(\frac{d\epsilon}{dt}\right) \qquad \Leftarrow \quad \frac{d\epsilon}{dt} < 0$$

for an initial condition of $\epsilon = \epsilon_1 \quad @ \quad t = 0$ we obtain

$$\frac{\epsilon(t)\epsilon_\infty}{\epsilon_i - \epsilon_\infty} = e^{-t/R\cdot C}$$

Heisler Charts

The lumped system analysis can be used if $Bi = hL/k < 0.1$ but what if $Bi > 0.1$

- need to solve the partial differential equation for temperature
- leads to an infinite series solution $\Rightarrow$ difficult to obtain a solution

The solution procedure for temperature is a function of several parameters

$$T(x,t) = f(x, L, t, k, \alpha, h, T_i, T_\infty)$$

By using dimensionless groups, we can reduce the temperature dependence to 3 dimensionless parameters

Dimensionless Group	**Formulation**
temperature	$\theta(x,t) = \dfrac{T(x,t) - T_\infty}{T_i - T_\infty}$
position	$x = x/L$
heat transfer	$Bi = hL/k$ Biot number
time	$Fo = \alpha t/L^2$ Fourier number

note: Cengel uses τ instead of Fo.

Now we can write

$$\theta(x,t) = f(x, Bi, Fo)$$

The characteristic length for the Biot number is

slab	$\mathcal{L} = L$
cylinder	$\mathcal{L} = r_o$
sphere	$\mathcal{L} = r_o$

contrast this versus the characteristic length for the lumped system analysis.

With this, two approaches are possible

1. use the first term of the infinite series solution. This method is only valid for $Fo > 0.2$
2. use the Heisler charts for each geometry as shown in Figs. 9-13, 9-14 and 9-15

First term solution: $Fo > 0.2 \rightarrow$ error about 2% max.

Plane Wall: $$\theta_{wall}(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \cos(\lambda_1, x/L)$$

Cylinder: $$\theta_{cyl}(r,t) = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \mathrm{J}_0(\lambda_1, r/r_o)$$

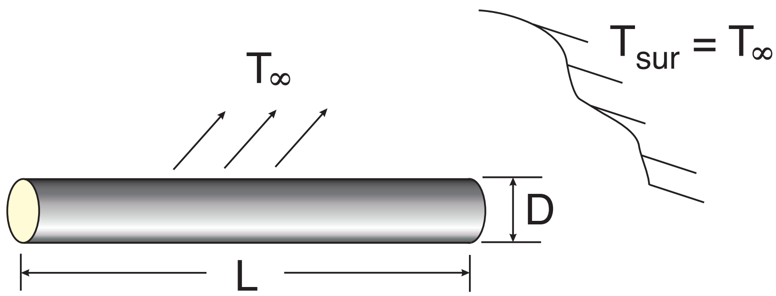
Sphere: $$\theta_{sph}(r,t) = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \frac{\sin(\lambda_1, r/r_o)}{\lambda_1, r/r_o}$$

where λ_1, A_1 can be determined from Table 9-1 based on the calculated value of the Biot number (will likely require some interpolation).

Heisler Charts

- find T_0 at the center for a given time
- find T at other locations at the same time
- find Q_{tot} up to time t

Example: **A fuse element**



Given:

$$D = 0.1\,mm \qquad T_{melt} = 900\,°C \qquad k = 20\,W/mK$$

$$L = 10\,mm \qquad T_\infty = 30\,°C \qquad \alpha = 5 \times 10^{-5}\,m^2/s \equiv k/\rho C_p$$

Assume:

- constant resistance $\mathcal{R} = 0.2\ ohms$
- the overall heat transfer coefficient is $h = h_{conv} + h_{rad} = 10\ W/m^2K$
- neglect any conduction losses to the fuse support

Find: the time t for the fuse to melt if a current of $I = 3\ amps$ suddenly starts to flow through the fuse

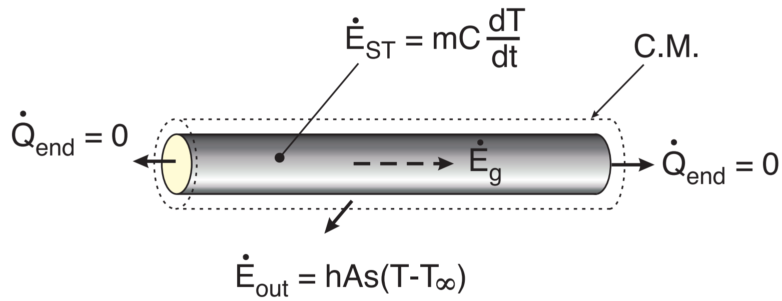
Solution: First, check the size of the Biot number

$$Bi = \frac{hV}{kA_s} = \frac{hD}{4k} = 1.25 \times 10^{-5} \ << 0.1$$

Therefore the lumped system approach is applicable and we can approximate $T \approx T(t)$ only.

Performing an energy balance over the fuse

$$\frac{dE_{ST}}{dt} = \dot{E}_{in} \nearrow^0 - \dot{E}_{out} + \dot{E}_g$$



where

$$\frac{dE_{ST}}{dt} = m\,C\,dT/dt$$

$$\dot{E}_{out} = hA_s(T - T_\infty)$$

$$\dot{E}_g = I^2\mathcal{R}$$

Therefore

$$m\,C\,\frac{dT}{dt} = -hA_s(T - T_\infty) + I^2\mathcal{R}$$

and

$$\frac{dT}{dt} = \underbrace{-\frac{hA_s}{m\,C}}_{-1/\tau}\left[\underbrace{T - T_\infty - \frac{I^2\mathcal{R}}{hA_s}}_{\theta(t)}\right]$$

Collecting terms

$$\frac{d\theta}{\theta} = -\frac{1}{\tau}\,dt \quad \Rightarrow \quad \ln\theta = -\frac{t}{\tau} + C_1$$

The initial conditions are

$$@t = 0 \quad \Rightarrow \quad T = T_i \qquad \Rightarrow \qquad C_1 = \ln\left(T_i - T_\infty - \frac{I^2\mathcal{R}}{hA_s}\right)$$

Therefore

$$T(t) = T_\infty + (T_i - T_\infty)e^{-t/\tau} + \frac{I^2\mathcal{R}}{hA_s}\left(1 - e^{-t/\tau}\right)$$

Using $T_i = T_\infty$ and $T(t) = T_{melt}$ we can determine the time, t for the fuse to blow out

$$900\,^\circ C = 30\,^\circ C + 5.73 \times 10^4\,^\circ C(1 - e^{-t})$$

Solving for t gives

$$t = 15.3\,ms \quad \Leftarrow$$