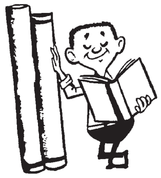
Radiation Heat Transfer



Reading	**Problems**
12-1 → 12-8	12-19, 12-29, 12-33, 12-47, 12-48, 12-58, 12-67, 12-76, 12-79, 12-83, 12-105

Introduction

Radiation is a photon emission that occurs when electrons change orbit. *Thermal radiation* occurs when the excitation is caused by heating.

How does the sun's energy travel to earth through a nearly perfect vacuum? Both conduction and convection required particle motion to produce heat transfer. The answer is *radiation*.

It should be readily apparent that radiation heat transfer calculations required several additional considerations in addition to those of conduction and convection, i.e.

- optical aspects: the manner in which an emitting body "sees" its neighbors
- surface conditions

Two different theories are available for describing the manner in which the radiation emitted from a surface propagates through the surrounding medium.

Theory 1: Maxwell's Electromagnetic Theory

Energy is transferred in the form of electromagnetic waves that travel at the speed of light.

$$C = \lambda \nu$$

where

$$C \;=\; \text{speed of light, } m/s$$

$$C_0 \;=\; 2.998 \times 10^8 \; [m/s] \quad \text{(vacuum conditions)}$$

$$\lambda \;=\; \text{wavelength, } m$$

$$\nu \;=\; \text{frequency, } Hz$$

Refraction, reflection and lense effects are all described by this theory

Theory 2: Einstein's Quantum Theory (1905)

Einstein made the remarkable assumption that light energy travelled in discrete packets called photons. Each photon had an energy

$$e = h\nu$$

where

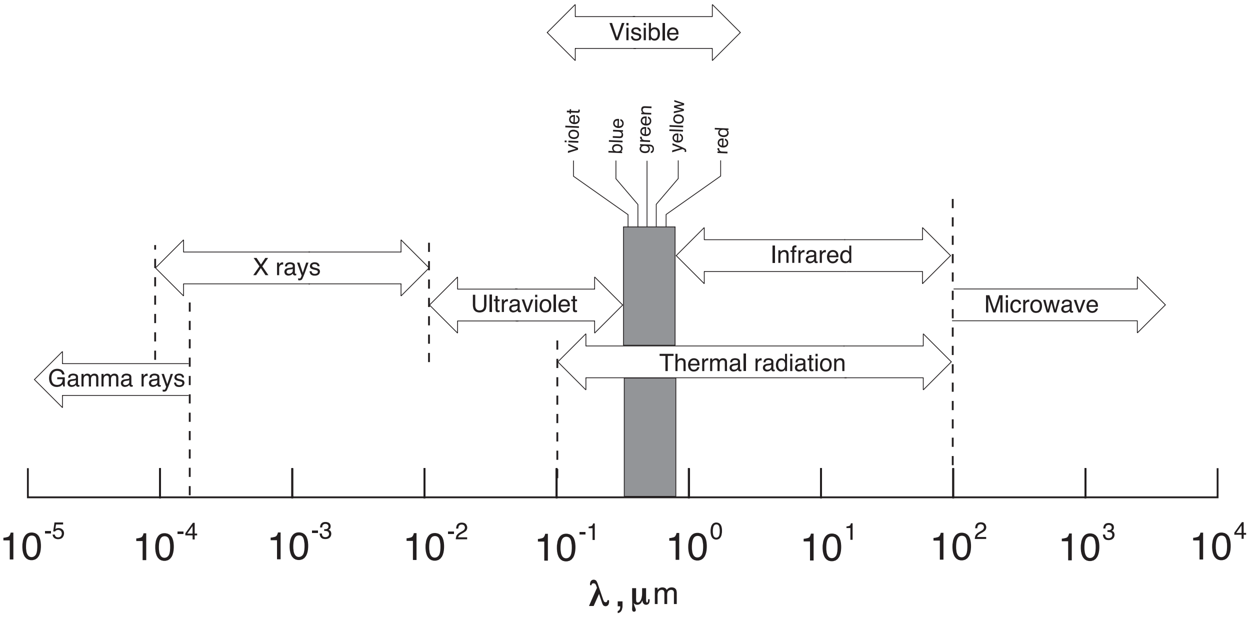
e = energy of a photon, J

h = Plank's constant $= 6.63 \times 10^{-34}\ J \cdot s$

ν = frequency, Hz

The idea that energy is discrete is a reasonable extension of the idea that matter is non-continuous at the molecular level.

The following figure shows the relatively narrow band occupied by thermal radiation.



An even narrower band inside the thermal radiation spectrum is denoted as the visible spectrum, that is the thermal radiation that can be seen by the human eye. The visible spectrum occupies roughly $0.4 - 0.7\ \mu m$. Thermal radiation is mostly in the infrared range. As objects heat up, their energy level increases, their frequency, ν, increases and the wavelength of the emitted radiation decreases. That is why objects first become red when heated and eventually turn white upon further heating.

Blackbody Radiation

A blackbody is an ideal radiator that

- *absorbs* all incident radiation regardless of wavelength and direction
- at a given temperature and wavelength, no surface can emit more energy than a blackbody
- *emitted* radiation is a function of wavelength and temperature but is independent of direction, i.e. a black body is a diffuse emitter (independent of direction)

Definitions

1. **Blackbody emissive power:** the radiation emitted by a blackbody per unit time and per unit surface area

$$E_b = \sigma\, T^4 \quad [W/m^2] \qquad\qquad \Leftarrow \text{ Stefan-Boltzmann law}$$

where

$$\sigma = \text{Stefan-Boltzmann constant } = 5.67 \times 10^{-8}\ W/(m^2 \cdot K^4)$$

and the temperature T is given in K.

2. **Spectral blackbody emissive power:** the amount of radiation energy emitted by a blackbody per unit surface area and per unit wavelength about the wavelength λ. The following relationship between emissive power, temperature and wavelength is known as *Plank's distribution law*

$$E_{b,\lambda} = \frac{C_1}{\lambda^5[\exp(C_2/\lambda T) - 1]} \quad [W/(m^2 \cdot \mu m)]$$

where

$$\begin{aligned}
C_0 &= 2.998 \times 10^8 \; [m/s] \quad \text{(vacuum conditions)} \\
C_1 &= 2\pi h C_0^2 = 3.743 \times 10^8 \; [W \cdot \mu m^4/m^2] \\
C_2 &= hC_0/K = 1.439 \times 10^4 \; [\mu \cdot K] \\
K &= \text{Boltzmann constant} \equiv 1.3805 \times 10^{-23} \; [J/K] \\
h &= \text{Plank's constant} \equiv 6.63 \times \; 10^{-34} \; [J \cdot s] \\
E_{b,\lambda} &= \text{energy of radiation in the wavelength} \\
& \quad\; \text{band } d\lambda \text{ per unit area and time}
\end{aligned}$$

If we integrate the spectral emissive power between $d\lambda = \lambda_2 - \lambda_1$ we will obtain the blackbody emissive power given as $E_b(T) = \sigma T^4$.

The wavelength at which the peak emissive power occurs for a given temperature can be obtained from *Wien's displacement law*

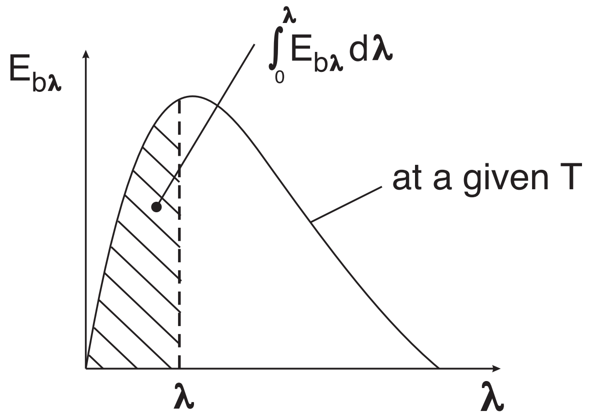
$$(\lambda T)_{max\ power} = 2897.8 \; \mu m \cdot K$$

- for solar radiation: $\lambda_{max} \approx 0.5 \; \mu m$ and $T_{max} \approx 5800 \; K$
- for a blackbody at $1000 \; K$: $\lambda_{max} \approx 2.9 \; \mu m$ at this wavelength the surface would appear red to the eye

3. **Blackbody radiation function:** the fraction of radiation emitted from a blackbody at temperature, T in the wavelength band $\lambda = 0 \rightarrow \lambda$

$$f_{0\rightarrow\lambda} = \frac{\displaystyle\int_0^{\lambda} E_{b,\lambda}(T) \; d\lambda}{\displaystyle\int_0^{\infty} E_{b,\lambda}(T) \; d\lambda} = \frac{\displaystyle\int_0^{\lambda} \frac{C_1}{\lambda^5[\exp(C_2/\lambda T) - 1]} d\lambda}{\sigma T^4}$$

let $t = \lambda T$ and $dt = T \; d\lambda$, then



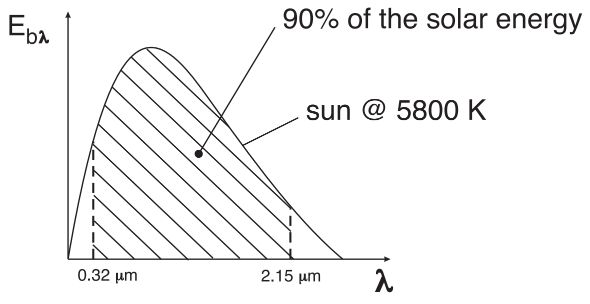
$$
\begin{aligned}
f_{0\to\lambda} &= \frac{\displaystyle\int_0^\lambda \frac{C_1 T^5 (1/T) dt}{t^5[\exp(C_2/t) - 1]}}{\sigma T^4} \\
&= \frac{C_1}{\sigma}\int_0^{\lambda T} \frac{dt}{t^5[\exp(C_2/t) - 1]} \\
&= f(\lambda T)
\end{aligned}
$$

$f_{0\to\lambda}$ is tabulated as a function λT in Table 12.2

We can easily find the fraction of radiation emitted by a blackbody at temperature T over a discrete wavelength band as

$$
\begin{aligned}
f_{\lambda_1\to\lambda_2} &= f(\lambda_2 T) - f(\lambda_1 T) \\
f_{\lambda\to\infty} &= 1 - f_{0\to\lambda}
\end{aligned}
$$

Example: In which λ range (λ_1 to λ_2) does 90% of solar energy lie?



Leave the top and bottom $\mathbf{5\%}$ off

$$\underbrace{\boldsymbol{f}_{\mathbf{0.32\rightarrow2.15}}}_{\mathbf{\equiv0.9}} = \underbrace{\boldsymbol{f}_{\mathbf{0.0\rightarrow2.15}}}_{\mathbf{\equiv0.95}} - \underbrace{\boldsymbol{f}_{\mathbf{0.0\rightarrow0.32}}}_{\mathbf{\equiv0.05}}$$

Radiation Properties of Real Surfaces

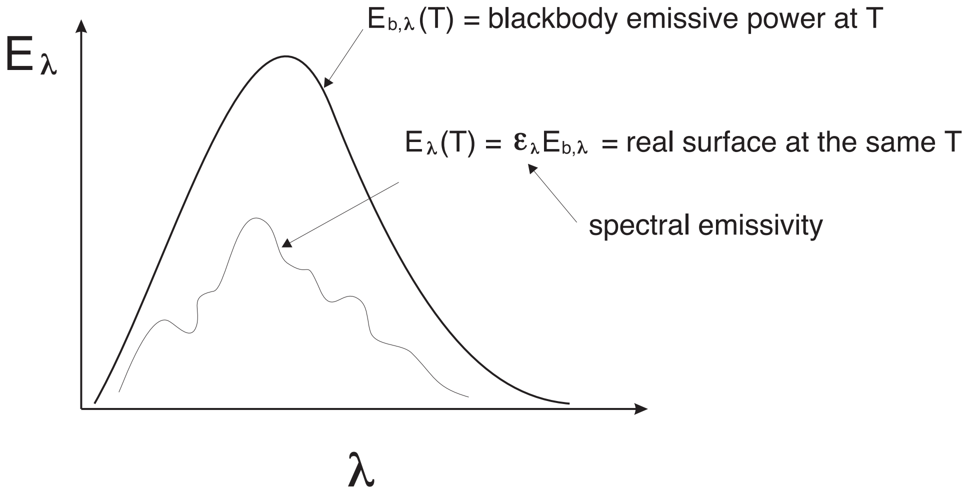
The thermal radiation emitted by a real surface is a function of surface temperature, $\boldsymbol{T}$, wavelength, $\boldsymbol{\lambda}$, direction and surface properties.

$$\boldsymbol{E_\lambda = f(T, \lambda, \textbf{direction}, \textbf{surface properties})} \quad \Rightarrow \text{spectral emissive power}$$

while for a blackbody, the radiation was only a function of temperature and wavelength

$$\boldsymbol{E_{b,\lambda} = f(T, \lambda)} \quad \rightarrow \quad \text{diffuse emitter} \quad \Rightarrow \text{independent of direction}$$

The following figure gives a comparison of the emissive power profile for a real surface and a blackbody.



Definitions

1. **Emissivity:** defined as the ratio of radiation emitted by a surface to the radiation emitted by a blackbody at the same surface temperature.

$$\begin{aligned}
\boldsymbol{\epsilon(T)} &= \frac{\text{radiation emitted by surface at temperature } \boldsymbol{T}}{\text{radiation emitted by a black surface at } \boldsymbol{T}} \\
&= \frac{\displaystyle\int_0^\infty \boldsymbol{E_\lambda(T)\, d\lambda}}{\displaystyle\int_0^\infty \boldsymbol{E_{b\lambda}(T)\, d\lambda}} = \frac{\displaystyle\int_0^\infty \boldsymbol{\epsilon_\lambda(T) E_{b\lambda}(T)\, d\lambda}}{\boldsymbol{E_b(T)}} = \frac{\boldsymbol{E(T)}}{\boldsymbol{\sigma T^4}}
\end{aligned}$$

where ϵ changes rather quickly with surface temperature.

Example: Dependence of ϵ on Temperature

$$\epsilon(T) = \frac{\int_0^\infty \epsilon_\lambda E_{b\lambda} d\lambda}{\sigma T^4} = 1.0 \times f_{1.5\to\infty}$$

Cold Surface

$$T = 300[K] \quad \lambda = 1.5\,\mu m \quad \lambda T = 450$$
$$f_{0\to 1.5} = 0.00 \quad f_{1.5\to\infty} = 1.0 \quad \Rightarrow \epsilon = 1.0$$

Very Hot Surface

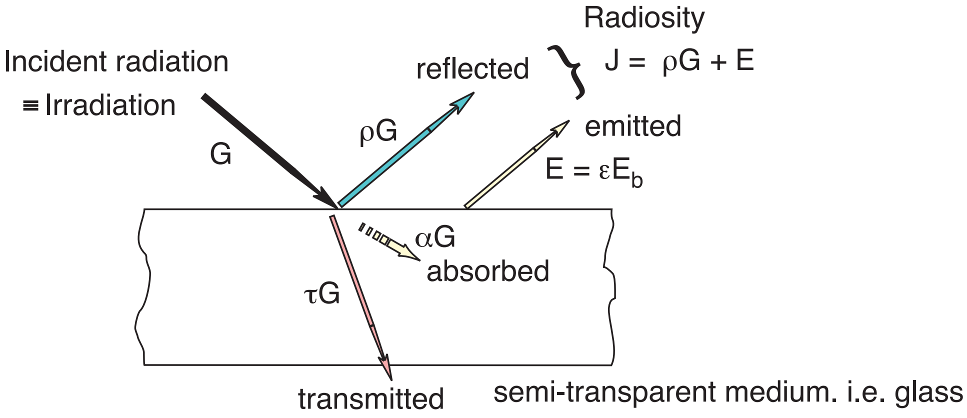
$$T = 4,000[K] \quad \lambda = 1.5\,\mu m \quad \lambda T = 6000$$
$$f_{0\to 1.5} = 0.738 \quad f_{1.5\to\infty} = 0.262 \quad \Rightarrow \epsilon = 0.262$$

The ϵ will vary with surface temperature because the distribution of radiation changes.

2. **Diffuse surface:** properties are independent of direction.

3. **Gray surface:** properties are independent of wavelength.

4. **Irradiation, G:** the radiation energy incident on a surface per unit area and per unit time



An energy balance based on incident radiation gives

$$G = \rho G + \alpha G + \tau G$$

where

$$\begin{aligned} G &= \text{incident radiation or irradiation, } W/m^2 \\ \rho G &= \text{reflected radiation, } W/m^2 \\ \alpha G &= \text{absorbed radiation, } W/m^2 \\ \tau G &= \text{transmitted radiation, } W/m^2 \end{aligned}$$

with the associated surface properties being

$$\left.\begin{aligned} \rho &= \text{reflectivity} \\ \alpha &= \text{absorptivity} \\ \tau &= \text{transmissivity} \end{aligned}\right\} \Rightarrow \text{ function of } \lambda \text{ \& } T \text{ of the incident radiation } G$$

$$\epsilon = \text{emissivity} \qquad \Rightarrow \quad \text{function of } \lambda \text{ \& } T \text{ of the emitting surface}$$

If we normalize with respect to the total irradiation

$$\alpha + \rho + \tau = 1$$

In general $\epsilon \neq \alpha$. However, for a diffuse-gray surface (properties are independent of wavelength and direction)

$$\epsilon = \alpha \qquad \text{diffuse-gray surface}$$

These unsubscripted values of α, ρ and τ represent the average properties, i.e. due to incident radiation energy from all directions over a hemispherical space and including all wavelengths.

We can just as easily define these properties for a specific wavelength, such that

$$G_\lambda = \rho_\lambda G_\lambda + \alpha_\lambda G_\lambda + \tau_\lambda G_\lambda$$

where

$$\begin{aligned} \rho_\lambda &= \text{spectral reflectivity } = f(\lambda, T) \\ \alpha_\lambda &= \text{spectral absorptivity } = f(\lambda, T) \\ \tau_\lambda &= \text{spectral transmissivity } = f(\lambda, T) \end{aligned}$$

and

$$\rho_\lambda + \alpha_\lambda + \tau_\lambda = 1$$

ϵ_λ depends strongly on the temperature of the emitting surface but not at all on the irradiation field G_λ.

5. **Radiosity, J:** the total radiation energy leaving a surface per unit area and per unit time.

   For a surface that is gray and opaque, i.e. $\epsilon = \alpha$ and $\alpha + \rho = 1$, the radiosity is given as

$$\begin{aligned} J &= \text{radiation emitted by the surface} + \text{radiation reflected by the surface} \\ &= \epsilon\, E_b + \rho G \\ &= \epsilon \sigma T^4 + \rho G \end{aligned}$$

Since $\rho = 0$ for a blackbody, the radiosity of a blackbody is

$$J = \sigma T^4$$

Diffuse-Gray Surfaces, $\epsilon = \alpha$

Kirchhoff's Law

The absorptivity, $\alpha(\lambda, T, \text{direction})$ *of a non-black surface is always equal to the emissivity,* $\epsilon(\lambda, T, \text{direction})$ *of the same surface when the surface is in thermal equilibrium with the radiation that impinges on it.*

$$\epsilon(\lambda, T, \phi, \theta) = \alpha(\lambda, T, \phi, \theta)$$

To a lesser degree of certainty we can write a more restrictive form of Kirchhoff's law for diffuse-gray surfaces where

$$\epsilon(T) = \alpha(T)$$

While Kirchhoff's law requires that the radiant source and the surface be in thermal equilibrium, this is seldom the case. The law can still be used but you should proceed with caution when the two temperatures differ by more than $100\ K$.

Conditions for $\epsilon = \alpha$

- in many radiation problems, the calculations are greatly simplified when $\epsilon = \alpha$, which defines a gray surface. Here, we seek the necessary conditions for $\epsilon = \alpha$. This is a 3-step procedure.
- the first step involves adopting *Kirchhoff's law* which is stated here without proof:

  $$\epsilon(\lambda, T, \phi, \theta) = \alpha(\lambda, T, \phi, \theta)$$

  This equation is always applicable because both $\epsilon(\lambda, T, \phi, \theta)$ and $\alpha(\lambda, T, \phi, \theta)$ are inherent surface properties
- the second step involves finding the requirements for $\epsilon(\lambda, T) = \alpha(\lambda, T)$. This is done using the definitions hemispherical radiation, as given below:

$$\left.\begin{aligned} \epsilon(\lambda, T) &= \frac{\displaystyle\int_0^{2\pi}\int_0^{\pi/2} \epsilon_{\lambda,\theta} \cos\theta \sin\theta d\theta d\phi}{\displaystyle\int_0^{2\pi}\int_0^{\pi/2} \cos\theta \sin\theta d\theta d\phi} \\ \alpha(\lambda, T) &= \frac{\displaystyle\int_0^{2\pi}\int_0^{\pi/2} \alpha_{\lambda,\theta} I_\lambda \cos\theta \sin\theta d\theta d\phi}{\displaystyle\int_0^{2\pi}\int_0^{\pi/2} I_\lambda \cos\theta \sin\theta d\theta d\phi} \end{aligned}\right\} \Rightarrow\ ???\quad \epsilon(\lambda, T) = \alpha(\lambda, T)$$

From this, we can see that for $\epsilon(\lambda, T) = \alpha(\lambda, T)$, we must have either a *diffuse surface* or a *diffuse irradiation*. As indicated before, in most engineering calculations, we use direction averaged properties which amounts to the assumption of a diffuse surface.

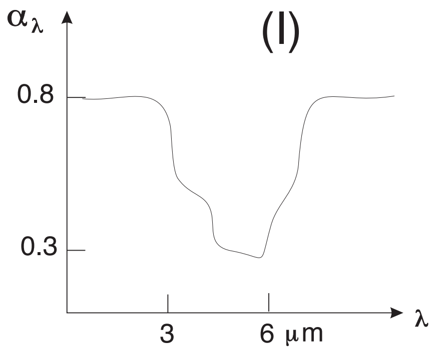
- having adopted $\epsilon(\lambda, T, \phi, \theta) = \alpha(\lambda, T, \phi, \theta)$ and $\epsilon(\lambda, T) = \alpha(\lambda, T)$, the third step involves finding the requirements for $\epsilon = \alpha$; that is:

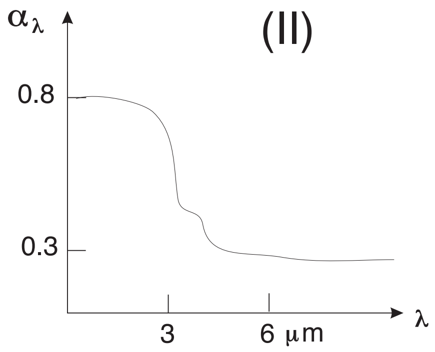
$$\epsilon(T) = \frac{\int_0^\infty \epsilon_\lambda E_{\lambda,b}\, d\lambda}{\int_0^\infty E_{\lambda,b}\, d\lambda} =?= \frac{\int_0^\infty \alpha_\lambda G_\lambda d\lambda}{\int_0^\infty G_\lambda d\lambda} = \alpha(T)$$

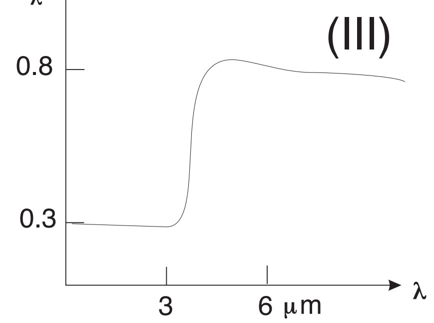
  From this, we can see that for $\epsilon(T) = \alpha(T)$, we must have either $G_\lambda = E_{\lambda,b}$ which means that the irradiation originated from a blackbody or $\epsilon(\lambda, T) = constant$ and $\alpha(\lambda, T) = constant$. To see the last point, remember that if $\epsilon(\lambda, T) = constant$ then $\epsilon(T) = \epsilon(\lambda, T)$ and similarly $\alpha(T) = \alpha(\lambda, T)$. But at step two, we have already established that $\epsilon(\lambda, T) = \alpha(\lambda, T)$, hence it follows that $\epsilon(T) = \alpha(T)$.

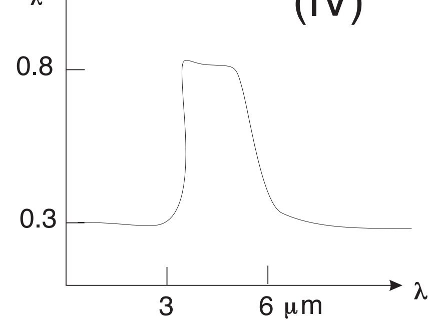
- by definition, a gray surface is one for which $\epsilon(\lambda, T)$ and $\alpha(\lambda, T)$ are independent of λ over the dominant spectral regions of G_λ and E_λ.

- for example: consider 4 different diffuse surfaces at the same temperature $T + 300\ K$ as shown below. These surfaces have been exposed to solar radiation. Which ones may be assumed to be gray?

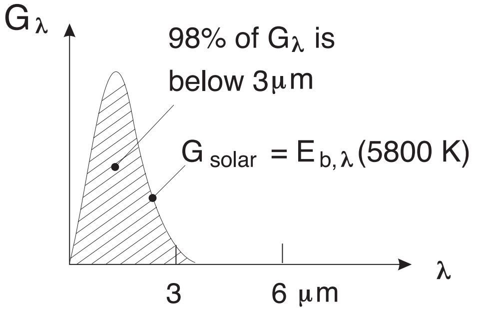


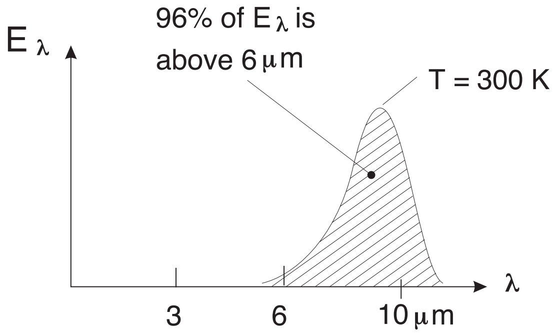






- for diffuse surfaces ⇒ $\alpha(\lambda, T) = \epsilon(\lambda, T)$
- based on the definition of a gray surface given above, we must determine the dominant spectral ranges of G_λ and E_λ These are shown below;





- by inspection we can see that:

Surface 1 : $\alpha = \epsilon = 0.8$, Gray surface

Surface 2 : $\alpha = 0.8$, $\epsilon = 0.3$, not a Gray surface

Surface 3 : $\alpha = 0.3$ $\epsilon = 0.8$, not a Gray surface

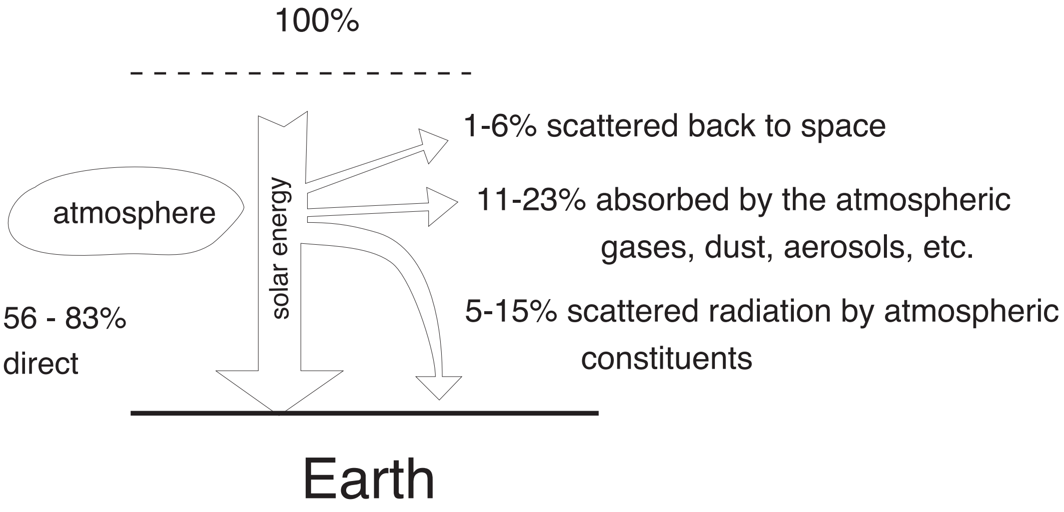
Surface 4 : $\alpha = \epsilon = 0.3$, Gray surface

Solar Radiation

The incident radiation energy reaching the earth's atmosphere is known as the solar constant, $\boldsymbol{G_s}$ and has a value of

$$\boldsymbol{G_s = 1353\ W/m^2}$$

While this value can change by about $\boldsymbol{\pm 3.4\%}$ throughout the year its change is relatively small and is assumed to be constant for most calculations. Although $\boldsymbol{G_s = 1353\ W/m^2}$ at the edge of the earth's atmosphere, the following figure shows how it is dispersed as it approaches the surface of the earth.



For most calculations we can use

$$\boldsymbol{E_{sky} \approx \sigma T_{sky}^4}$$

where the sky temperature is assumed to be

$$\boldsymbol{T_{sky} \approx 230\ K}\ \text{(cold, clear sky)} \ \longrightarrow\ \boldsymbol{285\ K}\ \text{(warm, cloudy sky)}$$

Emissions “from” the Earth’s surface

$$E = \epsilon \sigma T^4$$

where we can assume the earth’s surface temperature to be between $250 - 320\ K$ and

$$\begin{aligned} \epsilon_{water} &= 0.97 \\ \epsilon_{soil} &= 0.93 - 0.96 \\ \epsilon_{rocks} &= 0.88 - 0.95 \end{aligned}$$

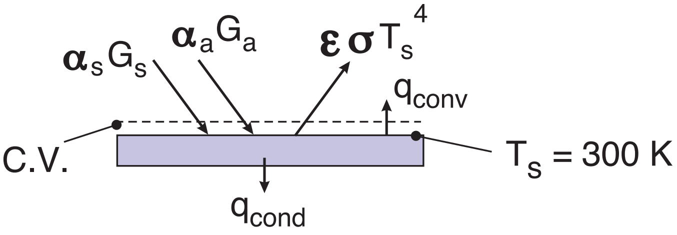
where all emissivities are evaluated at $300\ K$. On average we can use $\epsilon_{earth} \approx 0.95$

Atmospheric emission (mostly for CO_2 and H_2O)

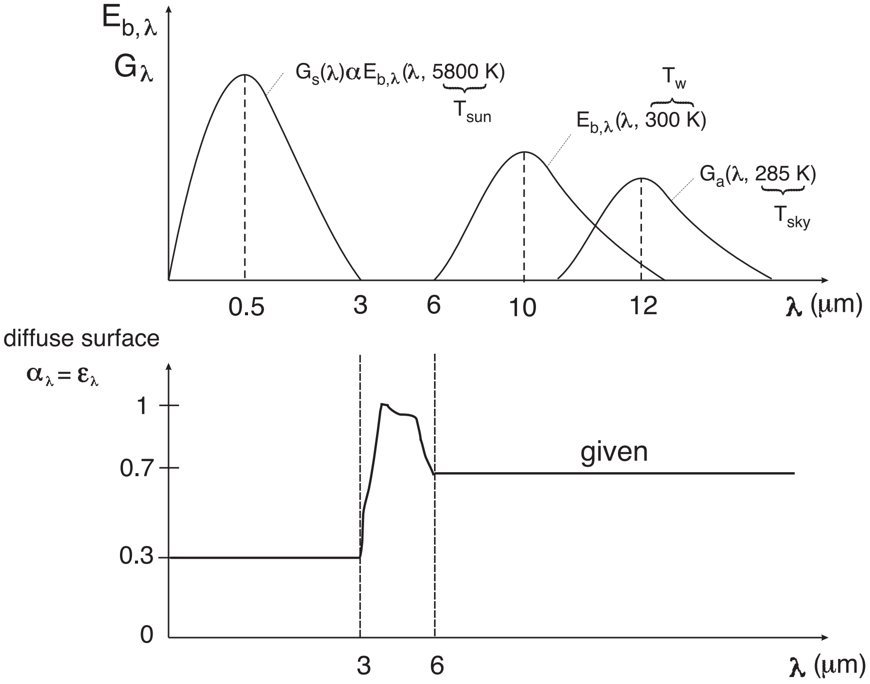
$$E_{sky} \approx \sigma T_{sky}^4$$

where T_{sky} is typically $230 - 285\ K$.

Example: **Diffuse surface exposed to solar and atmospheric irradiation**



Given:



Find: α_s, α_a, and ϵ

Solution: By definition

$$\alpha_s = \frac{\int_0^\infty \alpha_\lambda \, G_{s,\lambda} \, d\lambda}{\int_0^\infty G_{s,\lambda} \, d\lambda} = \frac{\int_0^\infty \alpha_\lambda \, E_{b,\lambda}(\lambda, 5800\ K) \, d\lambda}{\int_0^\infty E_{b,\lambda}(\lambda, 5800\ K) \, d\lambda}$$

$$= \frac{\cancelto{0}{\int_0^3} + \cancelto{0}{\int_3^6} + \int_6^\infty \alpha_\lambda \, E_{b,\lambda}(\lambda, 5800\ K) \, d\lambda}{\int_0^\infty E_{b,\lambda}(\lambda, 5800\ K) \, d\lambda}$$

Therefore

$$\alpha_s = \frac{\displaystyle\int_0^3 \alpha_\lambda \, E_{b,\lambda}(\lambda, 5800\ K)\, d\lambda}{\displaystyle\int_0^\infty E_{b,\lambda}(\lambda, 5800\ K)\, d\lambda} = 0.3\ f_{0\to 3\mu m}(5800\ K)$$

From Table 12-2 with $\lambda T = (3\ \mu m)(5800\ K) = 17,400\ \mu m \cdot K$, we find that

$$f_{0\to 3\ \mu m}(5800\ K) \approx 0.98$$

Therefore

$$\alpha_s = 0.3 \times 0.98 = 0.294 \qquad\qquad \alpha_{sun} = 0.294$$

$$\begin{aligned}
\alpha_a &= \frac{\displaystyle\int_0^\infty \alpha_\lambda \, G_{atm,\lambda}\, d\lambda}{\displaystyle\int_0^\infty G_{atm,\lambda}\, d\lambda} \\
&= \frac{\displaystyle\int_0^3 \cancelto{0}{\quad} + \int_3^6 \cancelto{0}{\quad} + \int_6^\infty \alpha_\lambda \, G_{atm,\lambda}\, d\lambda}{\displaystyle\int_0^\infty G_{atm,\lambda}\, d\lambda}
\end{aligned}$$

Since $G_{atm,\lambda} = E_{b,\lambda}(\lambda, 285\ K)$, then

$$\begin{aligned}
\alpha_a &= \frac{\displaystyle\int_6^\infty \alpha_\lambda \, E_{b,\lambda}(\lambda, 285\ K)\, d\lambda}{\displaystyle\int_0^\infty E_{b,\lambda}(\lambda, 285\ K)\, d\lambda} = 0.7\ f_{6\mu m\to\infty}(285\ K) \\
&= 0.7[1 - f_{0\to 6\mu m}(285\ K)]
\end{aligned}$$

For $\lambda T = (6\ \mu m)(285\ K) = 1710\ \mu\, m \cdot K$

$$f_{0\to 6\mu m}(285\ K) \approx 0.031$$

and

$$\alpha_a = 0.7[1 - 0.031] \approx 0.68 \qquad\qquad \Rightarrow \alpha_{atm} = 0.68$$

If we assume that the surface is diffuse

$$\epsilon \; = \; \frac{\int_0^\infty \epsilon_\lambda \, E_{b,\lambda}(\lambda, 300 \; K) \; d\lambda}{\int_0^\infty E_{b,\lambda}(\lambda, 300 \; K) \; d\lambda} = \frac{\int_0^\infty \alpha_\lambda \, E_{b,\lambda}(\lambda, 300 \; K) \; d\lambda}{\int_0^\infty E_{b,\lambda}(\lambda, 300 \; K) \; d\lambda}$$

Therefore

$$\epsilon = \frac{\int_6^\infty \alpha_\lambda \, E_{b,\lambda}(\lambda, 300 \; K) \; d\lambda}{\int_0^\infty E_{b,\lambda}(\lambda, 300 \; K) \; d\lambda} = 0.7 \; f_{6\mu m \rightarrow \infty}(300 \; K)$$

and

$$\epsilon = 0.7[1 - f_{0 \rightarrow 6\mu m}(300 \; K)]$$

Now for $\lambda T = (6 \; \mu m)(300 \; K) = 1800 \; \mu \cdot K$, we find from Table 12-2

$$\epsilon = 0.7 \times [1 - 0.042] \approx 0.67$$

Therefore

$$\epsilon \approx \alpha_{atm} \qquad \Rightarrow \text{ gray surface with respect to atmospheric irradiation}$$

but

$$\epsilon = 0.67 \; \neq \; \alpha_{sun} = 0.294$$

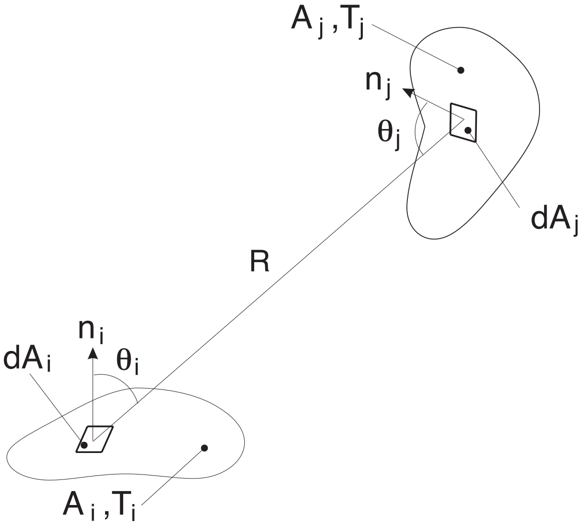
The surface is not gray with respect to solar irradiation.

View Factor (Shape Factor, Configuration Factor)

- the radiative exchange between surfaces clearly depends on how well the surfaces "see" one another. This information is provided by using shape factors (or view factors or configuration factors).
- **Definition:** The view factor, $F_{i \rightarrow j}$ is defined as the fraction of radiation leaving surface i which is intercepted by surface j. Hence

$$F_{i \rightarrow j} = \frac{\dot{Q}_{i \rightarrow j}}{A_i J_i} = \frac{\text{radiation reaching } j}{\text{radiation leaving } i}$$

- purely geometric, independent of surface properties and temperature
- based on the assumption that surfaces are diffuse emitters and reflectors
- for an N surface enclosure we should have N^2 view factors
- consider 2 surfaces with arbitrary size and configuration with respect to one another, which are diffuse surfaces.



It can be shown that

$$F_{i \rightarrow j} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j}{\pi R^2} dA_i dA_j$$

This is purely a geometrical property.

It is also found that

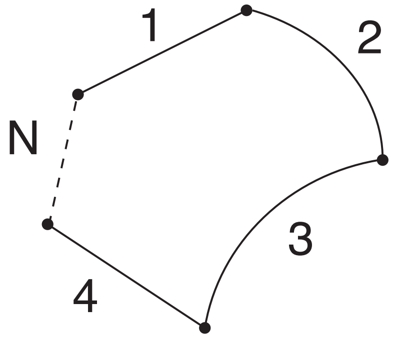
$$F_{j \rightarrow i} = \frac{1}{A_j} \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j}{\pi R^2} dA_i dA_j$$

The last two equations show that

$$A_i F_{i \rightarrow j} = A_j F_{j \rightarrow i}$$

This is called the reciprocity relation.

- consider an enclosure with N surfaces



Since this is an enclosure, the energy leaving a given surface is intercepted by the remaining surfaces in proportion to how well they "see" that surface. For example:

$$A_1 J_1 = \dot{Q}_{1 \rightarrow 1} + \dot{Q}_{1 \rightarrow 2} + \ldots + \dot{Q}_{1 \rightarrow N}$$

Therefore

$$1 = \sum_{j=1}^{N} \left(\frac{\dot{Q}_{i \rightarrow j}}{A_i J_i} \right) = \sum_{j=1}^{N} F_{i \rightarrow j}$$

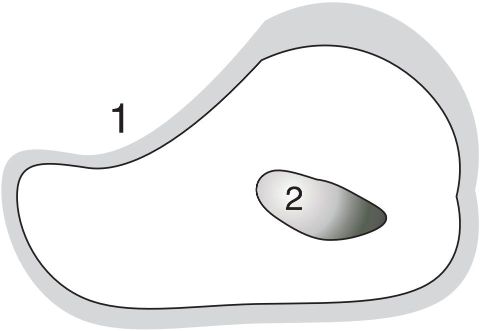
Hence

$$\sum_{j=1}^{N} F_{i \rightarrow j} = 1 \qquad ; \; i = 1, 2, \ldots, N$$

This is called the *summation rule*.

Note that $F_{i \rightarrow i} \neq 0$ for a concave surface. For a plane or convex surface $F_{i \rightarrow i} = 0$.

EXAMPLE:



By inspection $F_{2\rightarrow 1} = 1$ and $F_{2\rightarrow 2} = 0$. Using reciprocity:

$$A_1 F_{1\rightarrow 2} = A_2 F_{2\rightarrow 1} \quad \Rightarrow \quad F_{1\rightarrow 2} = A_2/A_1 F_{2\rightarrow 1}$$

Using the summation rule

$$F_{1\rightarrow 1} + F_{1\rightarrow 2} = 1 \quad \Rightarrow \quad F_{1\rightarrow 1} = 1 - A_2/A_1 F_{2\rightarrow 1}$$

- extensive compilations of shape factors exist. $F_{i\rightarrow j}$ for most common geometries can be found in any heat transfer text. See Tables 12-4 and 12.5 or Figures 12.41 - 12.44.

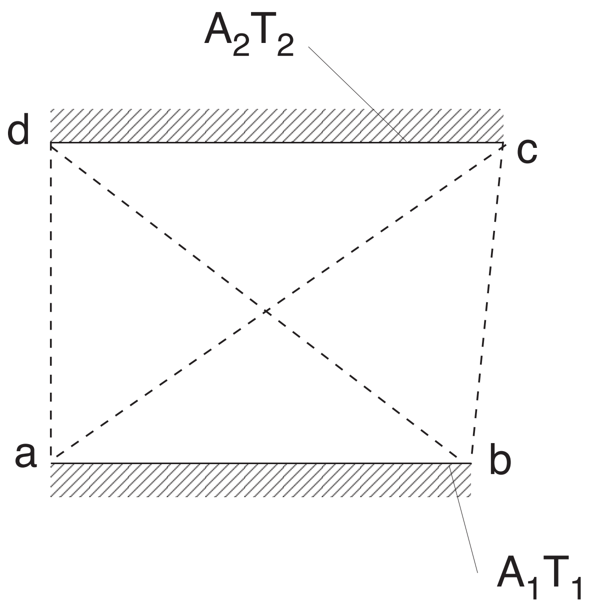
Hottel Crossed String Method

Can be applied to 2D problems where surfaces are any shape, flat, concave or convex. Note for a 2D surface the area, A is given as a length times a unit width.

$$A_1 F_{12} = A_2 F_{12} = \frac{\text{(total crossed)} - \text{(total uncrossed)}}{2}$$

A_1 and A_2 do not have to be parallel

$$A_1 F_{12} = A_2 F_{21} = \frac{1}{2}[\underbrace{(ac + bd)}_{crossed} - \underbrace{(bc + ad)}_{uncrossed}]$$



Proof:

$$\begin{aligned} A_1F_{1bd} &= \frac{ab+bd-ad}{2} \\ A_1F_{1ad} &= \frac{ab+ad-bd}{2} \\ A_1F_{1ac} &= \frac{ab+ac-bc}{2} \\ A_1F_{1bc} &= \frac{ab+bc-ac}{2} \end{aligned}$$

but for an enclosure we have

$$A_1F_{1ad} + A_1F_{1bc} + A_1F_{12} = A_1$$

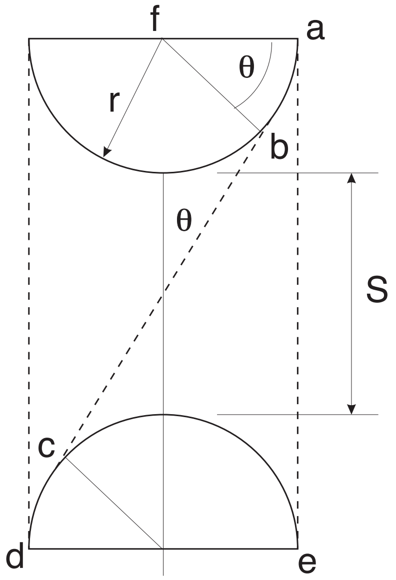
$$A_1F_{12} = A_1 - \{A_1F_{1ad} + A_1F_{1bc}\} \qquad (1)$$

Substituting F_{1ad} and F_{1bc} into Eq. (1) yields

$$\begin{aligned} A_1F_{12} &= ab - \left\{\frac{ab+ad-bd}{2} + \frac{ab+bc-ac}{2}\right\} \\ &= \frac{2ab-ab-ad+bd-ab-bc+ac}{2} \end{aligned}$$

$$= \frac{(ac+bd)-(bc+ad)}{2} \quad \Leftarrow \text{ cross strings formula}$$

Two Half Cylinders



By symmetry

$$\begin{aligned} ab &= cd = r\theta \\ \theta &= \sin^{-1}\frac{fb}{fo} = \sin^{-1}\frac{r}{r+s/2} \\ bo &= oc = \sqrt{(fo)^2-(fb)^2} = \sqrt{(r_s/2)^2-r^2} \end{aligned}$$

Let $x = 1 + s/(2r)$; $\theta = \sin^{-1} 1/x$; $bo = r\sqrt{x^2-1}$

$$ae = s + 2r = 2r\left(1+\frac{s}{2r}\right) = 2rx$$

Therefore

$$A_1F_{12} = A_2F_{21} = ad - ae = \overbrace{2r\sin^{-1}\frac{1}{x}}^{2r\theta} + 2r\sqrt{x^2-1} - 2rx$$

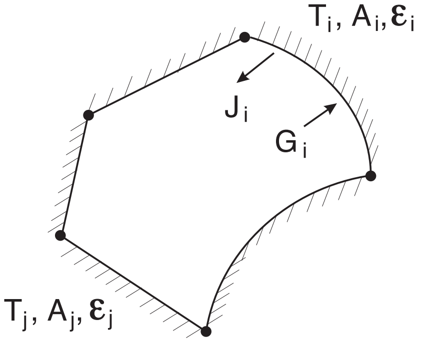
Since $\boldsymbol{A_1 = \pi r}$

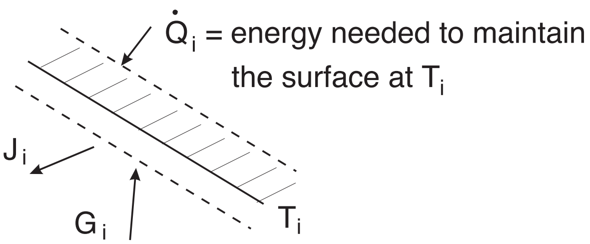
$$\boldsymbol{F_{12} = \frac{2}{\pi}\left\{\sqrt{x^2 - 1} + \sin^{-1}\frac{1}{x} - x\right\}}$$

Radiation Exchange Between Diffuse-Gray Surfaces Forming an Enclosure

We will assume that:

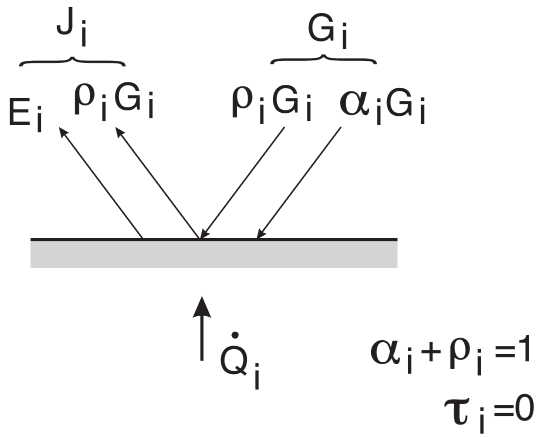
1. each surface of the enclosure is isothermal
2. radiosity, J_i, and irradiation, G_i are uniform over each surface
3. the surfaces are opaque ($\tau_i = 0$) and diffuse-gray ($\alpha_i = \epsilon_i$)
4. the cavity is filled with a fluid which does not participate in the radiative exchange process





- an energy balance on the $i'th$ surface gives:

$$\dot{Q}_i = \dot{q}_i A_i = A_i(J_i - G_i)$$



$$\dot{Q}_i = A_i(E_i - \alpha_i G_i) \qquad (1)$$

$$J_i = E_i + \rho_i G_i \qquad (2)$$

$$E_i = \epsilon_i E_{b,i} = \epsilon_i \sigma T_i^4 \qquad (3)$$

$$\rho_i = 1 - \alpha_i = 1 - \epsilon_i \qquad (4) \qquad \Rightarrow \text{since } \alpha_i + \rho_i + \cancelto{0}{\tau_i} = 1$$

$$\text{and} \quad \alpha_i = \epsilon_i$$

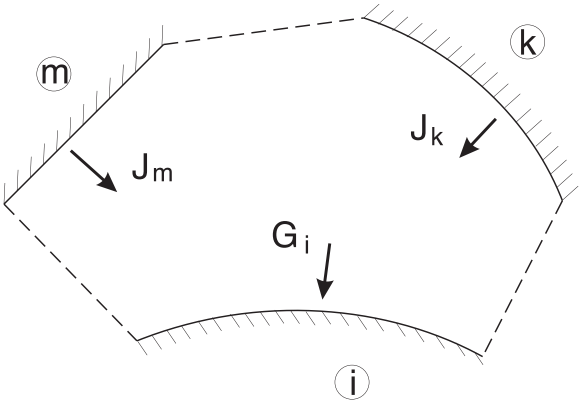
Combining Eqs. 2, 3 and 4 gives

$$J_i = \epsilon_i E_{b,i} + (1 - \epsilon_i) G_i$$

Combining this with Eq. 1 gives

$$Q_i = \frac{E_{b,i} - J_i}{\left(\dfrac{1-\epsilon_i}{\epsilon_i A_i}\right)} \equiv \frac{\text{potential difference}}{\text{surface resistance}}$$

This surface resistance represents real surface behavior as opposed to a blackbody surface. Recall that for a blackbody $\epsilon_i = \alpha_i = 1$ and $J_i = E_1 = E_{b,i} = \sigma T_i^4$.



- next consider radiative exchange between the surfaces.

By inspection it is clearly seen that

$$\left\{ \begin{array}{c} \text{irradiation on} \\ \text{surface } i \end{array} \right\} = \left\{ \begin{array}{c} \text{radiation leaving the} \\ \text{remaining surfaces} \end{array} \right\}$$

$$A_i G_i = \sum_{j=1}^{N} F_{j \to i}(A_j J_j) = \sum_{j=1}^{N} A_i F_{i \to j} J_i$$

Therefore

$$\boldsymbol{G_i = \sum_{j=1}^{N} F_{i\rightarrow j} J_j}$$

Combining this with Eqs. 2 - 4 gives

$$\boldsymbol{J_i = \epsilon_i \sigma T_i^4 + (1 - \epsilon_i) \sum_{j=1}^{N} F_{i\rightarrow j} J_j}$$

In addition we can write

$$\boldsymbol{\dot{Q}_i = A_i J_i - \sum_{j=1}^{N} A_i F_{i\rightarrow j} J_j}$$

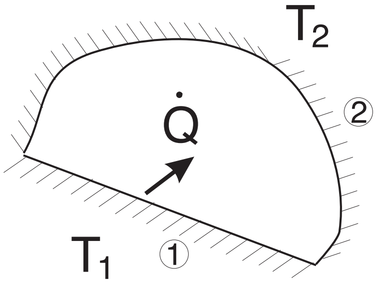
Since the summation rule states $\boldsymbol{\sum_{j=1}^{N} F_{i\rightarrow j} = 1}$, the above equation becomes

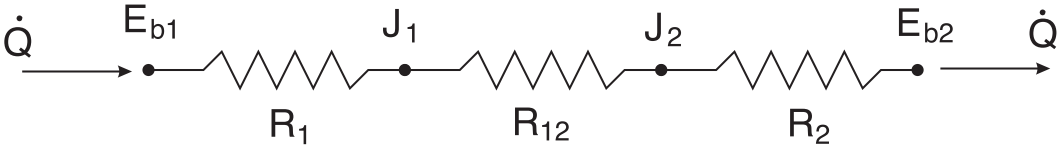
$$\begin{aligned} \boldsymbol{\dot{Q}_i} &= \boldsymbol{A_i \left\{ \sum_{j=1}^{N} F_{i\rightarrow j} J_i - \sum_{j=1}^{N} F_{i\rightarrow j} J_j \right\}} \\ \boldsymbol{\dot{Q}_i} &= \boldsymbol{\sum_{j=1}^{N} A_i F_{i\rightarrow j} (J_i - J_j) = \sum_{j=1}^{N} \dot{Q}_{i\rightarrow j}} \end{aligned}$$

or

$$\boldsymbol{\dot{Q}_i = \sum_{j=1}^{N} \frac{J_i - J_j}{\left(\dfrac{1}{A_i F_{i\rightarrow j}} \right)}} \equiv \frac{\text{potential difference}}{\text{configuration resistance}}$$

EXAMPLE: Two-surface enclosures





where

$$\begin{aligned} \boldsymbol{R_1} &= \left(\frac{\boldsymbol{1-\epsilon_1}}{\boldsymbol{\epsilon_1 A_1}}\right) \\ \boldsymbol{R_{12}} &= \left(\frac{\boldsymbol{1}}{\boldsymbol{A_1 F_{1\to 2}}}\right) \\ \boldsymbol{R_2} &= \left(\frac{\boldsymbol{1-\epsilon_2}}{\boldsymbol{\epsilon_2 A_2}}\right) \end{aligned}$$

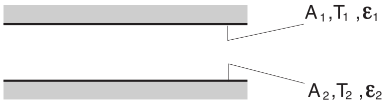
Clearly

$$\boldsymbol{\dot{Q} = \frac{(E_{b,1} - E_{b,2})}{\sum R}} \qquad ; \quad \boldsymbol{E_b = \sigma T^4}$$

Some very useful consequences of this equation are given below.

Special Diffuse, gray, two-surface enclosures

Large (infinite) Parallel Plates

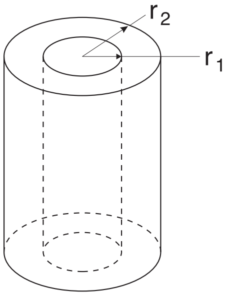


$$\boldsymbol{A_1 = A_2 = A}$$

$$\boldsymbol{F_{1\rightarrow 2} = 1}$$

$$\boldsymbol{\dot{Q}_{12} = \frac{A\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_2} - 1}}$$

Long (infinite) Concentric Cylinders

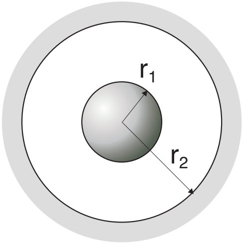


$$\boldsymbol{\frac{A_1}{A_2} = \frac{r_1}{r_2}}$$

$$\boldsymbol{F_{1\rightarrow 2} = 1}$$

$$\boldsymbol{\dot{Q}_{12} = \frac{A\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\epsilon_1} + \dfrac{1-\epsilon_2}{\epsilon_2}\left(\dfrac{r_1}{r_2}\right)}}$$

Concentric Spheres

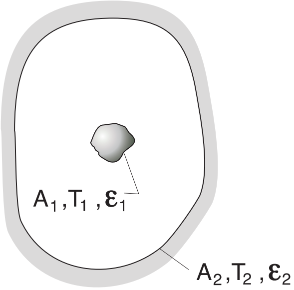


$$\boldsymbol{\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}}$$

$$\boldsymbol{F_{1\rightarrow 2} = 1}$$

$$\boldsymbol{\dot{Q}_{12} = \frac{A\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\epsilon_1} + \dfrac{1-\epsilon_2}{\epsilon_2}\left(\dfrac{r_1}{r_2}\right)^2}}$$

Small Convex Object in a Large Cavity

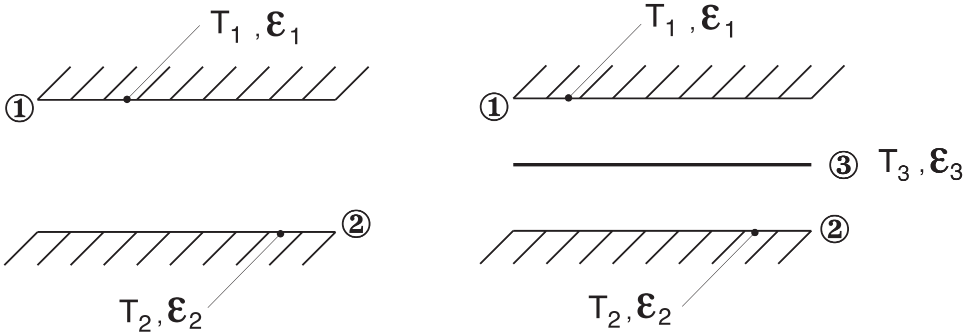


$$\frac{A_1}{A_2} = 0$$

$$\dot{Q}_{12} = \sigma A_1 \epsilon_1 (T_1^4 - T_2^4)$$

$$F_{1\to 2} = 1$$

Example: Radiation between two large parallel plates



CASE 1

CASE 2

Given:

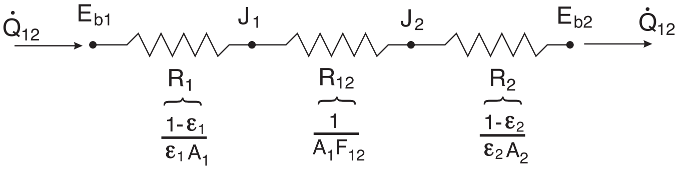
$$\begin{aligned} \epsilon_1 &= 0.2 & T_1 &= 800\ K \\ \epsilon_2 &= 0.7 & T_2 &= 500\ K \\ \epsilon_3 &= 0.02 & A_1 &= A_2 = A_3 = A \end{aligned}$$

Assume steady state conditions.

Find: $\dot{Q}_{12}/A$ for both cases and T_3 for Case 2.

Solution:

Case 1: The radiation network looks like



for the flat plate , $F_{11} = 0$ and from the summation rule

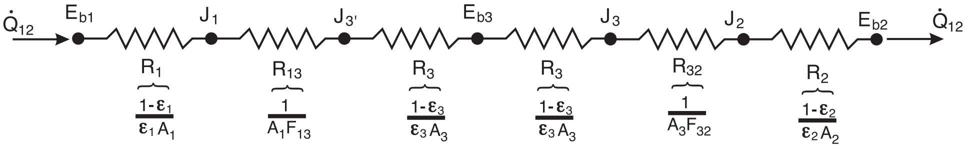
$$F_{\cancel{11}}^{\nearrow 0} + F_{12} = 1 \qquad \Rightarrow \qquad F_{12} = 1$$

Therefore

$$\dot{Q}_{12} = \frac{E_{b1} - E_{b2}}{R_1 + R_{12} + R_2} = \frac{A\sigma(T_1^4 - T_2^4)}{\dfrac{1-\epsilon_1}{\epsilon_1} + \dfrac{1}{1} + \dfrac{1-\epsilon_2}{\epsilon_2}}$$

$$
\begin{aligned}
\frac{\dot{Q}_{12}}{A} &= \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\epsilon_1} + + \dfrac{1}{\epsilon_2} - 1} = \frac{5.67 \times 10^{-8}\ W/(m^2 \cdot K^4)(800^4 - 500^4)\ K^4}{\dfrac{1}{0.2} + \dfrac{1}{0.7} - 1} \\
&= 3625\ W/m^2
\end{aligned}
$$

Case 2: Plate 3 is thin so that the conduction resistance is negligible. The temperature on both sides of surface 3 is the same, at T_3.



$$
\begin{aligned}
\dot{Q}_{12} &= \frac{E_{b1} - E_{b2}}{A \sum R} = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1-\epsilon_1}{\epsilon_1} + 1 + \dfrac{1-\epsilon_3}{\epsilon_3} + \dfrac{1-\epsilon_3}{\epsilon_3} + 1 \dfrac{1-\epsilon_2}{\epsilon_2}} \\
&= \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\epsilon_1} + 2 \times \left(\dfrac{1}{\epsilon_3} - 1\right) + \dfrac{1}{\epsilon_2} - 1} \\
&= \frac{5.67 \times 10^{-8}\ W/(m^2 \cdot K^4)(800^4 - 500^4)\ K^4}{\dfrac{1}{0.2} + 2 \times \left(\dfrac{1}{0.02} - 1\right) + \dfrac{1}{0.7} - 1} = 188.5\ W/m^2
\end{aligned}
$$

This is equivalent to a 94.8% reduction because of the thin film at 3 impeding the transfer of radiant energy. Plate 3 can be referred to as a radiation shield.

$$
\dot{Q}_{12} = \frac{E_{b1} - E_{b3}}{R_1 + R_{13} + R_3}
$$

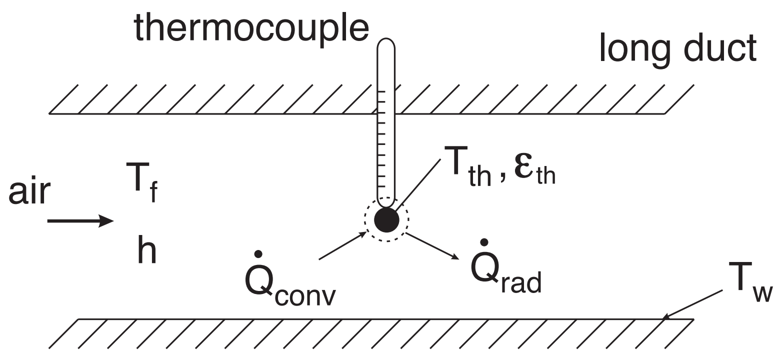
Therefore

$$
\underbrace{E_{b3}}_{\sigma T_3^4} = \underbrace{E_{b1}}_{\sigma T_1^4} - \dot{Q}_{12}(R_1 + R_{13} + R_3)
$$

and

$$\begin{aligned} \boldsymbol{T_3} &= \left\{ \boldsymbol{T_1^4} - \frac{\dot{\boldsymbol{Q}}_{\mathbf{12}}}{\boldsymbol{A}} \left(\frac{\mathbf{1} - \boldsymbol{\epsilon_1}}{\boldsymbol{\epsilon_1}} + \mathbf{1} + \frac{\mathbf{1} - \boldsymbol{\epsilon_3}}{\boldsymbol{\epsilon_3}} \right) \frac{\mathbf{1}}{\boldsymbol{\sigma}} \right\}^{\mathbf{1/4}} \\ &= \left\{ \mathbf{800^4} = \frac{\mathbf{188.5}}{\mathbf{5.67 \times 10^{-8}}} \left(\frac{\mathbf{1}}{\mathbf{0.2}} + \frac{\mathbf{1}}{\mathbf{0.02}} - \mathbf{1} \right) \right\}^{\mathbf{1/4}} \\ &= \mathbf{692.6}\ \boldsymbol{K} \end{aligned}$$

Example: Radiation effect on temperature measurements



Given:

$$\begin{array}{rcl@{\qquad\qquad}rcl} T_w & = & 400\ K & T_{th} & = & 650\ K \\ \epsilon_{th} & = & 0.6 & h & = & 80\ W/(m^2 \cdot K) \end{array}$$

Assume steady state conditions.

Find: T_f the air temperature

Solution: Assume conduction from the thermocouple junction to the thermocouple support is negligible.

An energy balance gives

$$\frac{d\cancelto{0}{E_{C.M.}}}{dt} = \dot{E}_{in} - \dot{E}_{out} + \cancelto{0}{\dot{E}_g}$$

Therefore

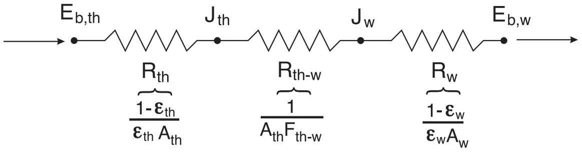
$$\dot{Q}_{conv} - \dot{Q}_{rad} = 0 \qquad (1)$$

From Newton's law of cooling we know

$$\dot{Q}_{conv} = hA_{th}(T_f - T_{th}) \qquad (2)$$

For radiation between the thermocouple and the duct wall we will assume $A_{th} << A_{wall}$ since the thermocouple area is very small in relation to the area of the wall.

The radiation network looks like



$$\begin{aligned} \dot{Q}_{rad} &= \frac{E_{b,th} - E_{b,w}}{\sum R} \\ &= \epsilon_{th} A_{th} \sigma (T_{th}^4 - T_w^4) \qquad (3) \end{aligned}$$

since $A_{th}/A_w \approx 0$

Substituting (2) and (3) into (1) gives

$$hA_{th}(T_f - T_{th}) - \epsilon_{th} A_{th} \sigma (T_{th}^4 - T_w^4) = 0$$

Therefore

$$\begin{aligned} T_f &= T_{th} + \frac{\epsilon_{th}\sigma}{h}(T_{th}^4 - T_w^4) \\ &= 650\ K + \frac{0.6 \times 5.67 \times 10^{-8}\ W/(m^2 \cdot K^4)}{80\ W/(m^2 \cdot K)}(650^4 - 400^4)\ K^4 \\ &= 650\ K + 65\ K \\ &= 715\ K \end{aligned}$$