University of Waterloo

ME201

ADVANCED CALCULUS
FINAL EXAMINATION

April 10, 2017 12:30 pm - 3:00 pm

Rooms: DWE 2402 & CPH 3679

Instructor: R. Culham

Name: ______________________________

Student ID Number: ______________________________

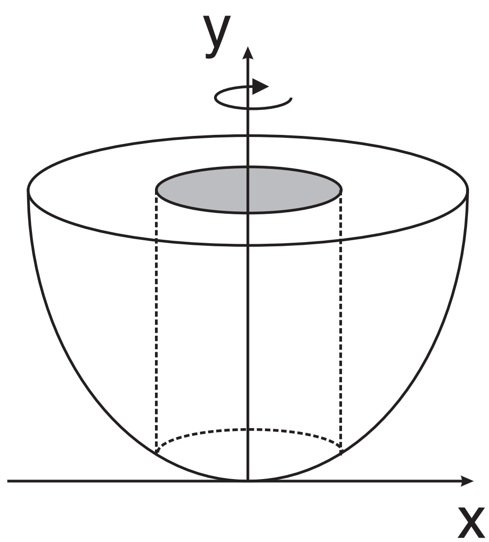
Instructions

1. This is a 2.5 hour, closed-book examination.
2. Permitted aids include:
   - one 8.5 in. $\times$ 11 in. crib sheet, (both sides)
   - Mathematical Handbook of Formulas and Tables, 4th ed., M.R. Spiegel, S. Lipschutz and J. Liu, Schaum's Outline Series, 2013.
   - calculator
3. Answer all questions in the space provided. If additional space is required, use the back of the pages or the blank pages included.
4. It is your responsibility to write clearly and legibly. Clearly state all assumptions. Part marks will be given for part answers, provided that your methodology is clear.

Question	**Marks**	**Grade**
1	9	
2	9	
3	10	
4	12	
5	15	
TOTAL	55	

Question 1 (9 marks)

A solid is generated by revolving the region bounded by $y = x^2/4$ and $y = 4$ about the $y-$axis. A circular cylindrical hole, centered along the axis of revolution, is drilled through this solid so that 25% of the volume is removed. Find the diameter of the hole.



Using the shell method, we can find the total volume of the paraboloid.

$$\text{shell} \quad \Rightarrow \quad 2\pi x dx \ \times \Delta y$$

$$\text{bounds on } y \quad \Rightarrow \quad \text{upper: } y = 4, \quad \text{lower: } y = x^2/4$$

$$\text{bounds on } x \quad \Rightarrow \quad 0 \leq x \leq 4$$

therefore the total volume is

$$\begin{aligned} V &= \int_{x=0}^{4} 2\pi x \Delta y \, dx \\ &= 2\pi \int_{x=0}^{4} x \left(4 - x^2/4\right) \, dx \\ &= 2\pi \int_{x=0}^{4} \left(4x - x^3/4\right) \, dx \\ &= 2\pi \left[2x^2 - \frac{x^4}{16}\right]_0^4 = 32\pi \end{aligned}$$

If a hole is drilled in the center with a radius of x_0 in order to remove $1/4$ of the volume, we have

$$\begin{aligned} 0.75V &= 2\pi \int_{x_0}^{4} x\left(4 - x^2/4\right)\, dx \\ 0.75 \times 32\pi &= 2\pi \left[2x^2 - \frac{x^4}{16}\right]_{x_0}^{4} \\ 24\pi &= 2\pi \left[(32-16) - \left(2x_0^2 - \frac{x_0^4}{16}\right)\right] \end{aligned}$$

Collecting terms we get

$$x_0^4 - 32x_0^2 + 64 = 0$$

$$\begin{aligned} x_0^2 &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{32 \pm \sqrt{32^2 - 4 \cdot 64}}{2} \\ &= 16 \pm 8\sqrt{3} \end{aligned}$$

Since x_0 must be between $0 \ \leq x_0 \leq 4$, we are not interested in $x_0 = \sqrt{16+8\sqrt{3}} \approx 5.46$

We accept

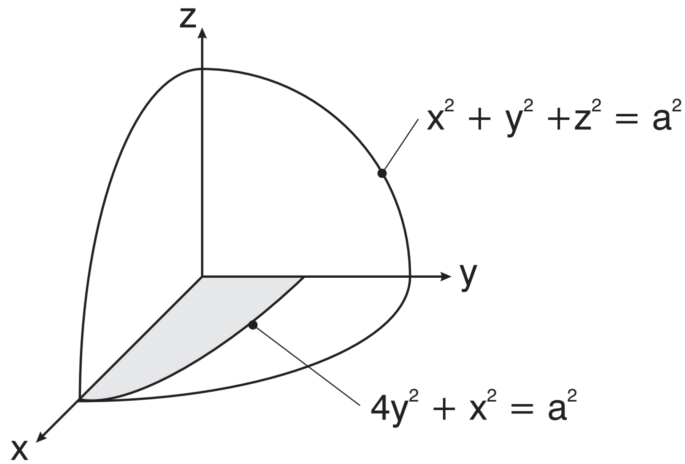
$$x_0 = \sqrt{16 - 8\sqrt{3}} \approx 1.464$$

and the diameter of the hole is

$$\text{diameter} \ = \ 2x_0 = 2.982 \Leftarrow$$

Question 2 (9 marks)

Determine the surface area in the positive $x, y, z-$ octant for the portion of the sphere, $x^2 + y^2 + z^2 = a^2$, that lies inside an elliptic cylinder centered about the $z-$axis, given as $4y^2 + x^2 = a^2$.



The surface of the sphere can be written in terms of z as

$$z = f(x, y) = (a^2 - x^2 - y^2)^{1/2}$$

The partial derivative of $f(x, y)$ with respect to x and y are

$$\frac{\partial f}{\partial x} = \frac{-2x}{2 \cdot (a^2 - x^2 - y^2)^{1/2}}$$

$$\frac{\partial f}{\partial y} = \frac{-2y}{2 \cdot (a^2 - x^2 - y^2)^{1/2}}$$

we can write the surface area as

$$\begin{aligned} S &= \int_{x=0}^{a} \int_{y=0}^{\frac{1}{2}\sqrt{a^2-x^2}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dy\, dx \\ &= \int_{x=0}^{a} \int_{y=0}^{\frac{1}{2}\sqrt{a^2-x^2}} \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}\, dy\, dx \\ &= a \int_{x=0}^{a} \int_{y=0}^{\frac{1}{2}\sqrt{a^2-x^2}} \frac{1}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx \end{aligned}$$

From Schaum's 17.11.1, if we hold x fixed and let $A^2 = a^2 - x^2$

$$\int \frac{dy}{\sqrt{A^2 - y^2}} = \sin^{-1}\frac{y}{A}$$

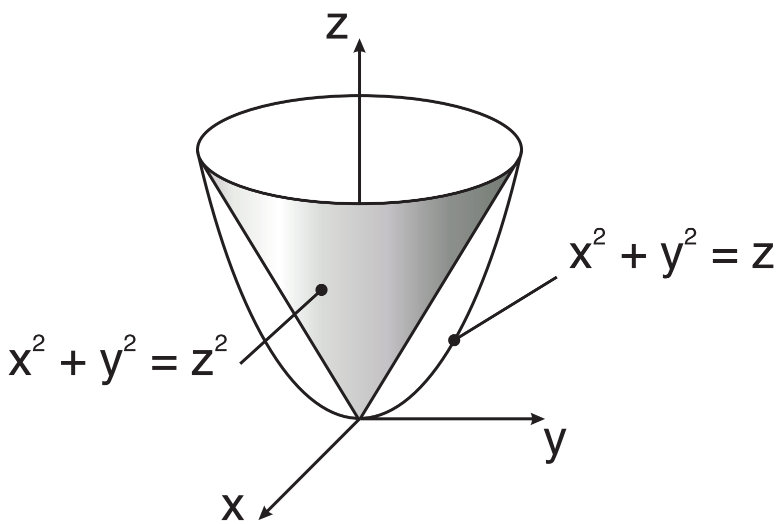
$$\begin{aligned} S &= a\int_{x=0}^{a} \sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\Bigg|_0^{\frac{1}{2}\sqrt{a^2-x^2}} dx \\ &= a\cdot\frac{\pi}{6}\Big[x\Big]_0^a \\ &= \frac{1}{6}\pi a^2 \Leftarrow \end{aligned}$$

Question 3 (10 marks)

Consider the region bounded between the inside of the paraboloid, $x^2 + y^2 = z$, and the outside of the cone $x^2 + y^2 = z^2$, as shown in the figure below.

a) find the center of mass of the solid region between the paraboloid and the cone for a homogeneous solid with a density, ρ.

b) calculate the moment of inertia about the $z-$axis.



Part a)

The triple integration is best solved in circular cylindrical coordinates where

$$M = \int\int\int \rho \, dV = \int\int\int \rho \, dz \, r \, dr \, d\theta$$

and the bounds of integration are:

z : upper bound $\rightarrow z = \sqrt{x^2 + y^2} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r$

lower bound $\rightarrow z = x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

r : find intersection of the cone and the paraboloid

$z^2 = z \quad \rightarrow z = 0$ or $z = 1$

upper bound $\rightarrow r = 1$

lower bound $\rightarrow r = 0$

θ : $0 \leq \theta \leq 2\pi$

Therefore

$$M = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r^2}^{r} \rho \, dz \, r \, dr \, d\theta$$

$$
\begin{aligned}
&= \rho\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(r-r^2)r\;dr\;d\theta \\
&= \rho\int_{\theta=0}^{2\pi} \frac{1}{3}r^3 - \frac{1}{4}r^4\Big|_0^1\;d\theta \\
&= \frac{\rho\pi}{6}
\end{aligned}
$$

From symmetry, it is clear that $\overline{x} = 0$ and $\overline{y} = 0$, and

$$
\begin{aligned}
\overline{z} &= \frac{\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r^2}^{r}\rho\;z\;dz\;r\;dr\;d\theta}{M} \\
&= \frac{(\rho\pi)/12}{(\rho\pi)/6} = \frac{1}{2}
\end{aligned}
$$

The center of mass is $\left(0, 0, \frac{1}{2}\right) \Leftarrow$

Part b)
The moment of inertia about the $z-$axis is

$$
\begin{aligned}
J_z &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r^2}^{r}\rho\;(x^2+y^2)\;dz\;r\;dr\;d\theta \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r^2}^{r}\rho\;r^3\;dz\;dr\;d\theta \\
&= \frac{\rho\pi}{15} \Leftarrow
\end{aligned}
$$

Question 4 (12 marks)

Given the vector function

$$\vec{F} = (3x^2yz)\,\hat{i} \;+\; (x^3z)\,\hat{j} \;+\; (x^3y - 4z)\,\hat{k}$$

i) determine if the field is independent of path
ii) if yes, find the scalar potential function, ϕ, such that $\vec{F} = \nabla\phi$
iii) find the work done ($Joules$) by the field if travelling along a path formed by the intersection of $y = 2x$ and $x^2 + y^2 + z^2 = 54$ from $(1, 2, 7)$ to $(3, 6, 3)$

Part i)

Check to see if $\nabla \times \vec{F} = 0$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2yz & x^3z & x^3y - 4z \end{vmatrix}$$

$$= (x^3 - x^3)\,\hat{i} \;-\; (3x^2y - 3x^2y)\,\hat{j} \;+\; (3x^2z - 3x^2z)\,\hat{k}$$

$$= 0 \qquad \Leftarrow \text{Conservative \& Independent of Path}$$

Part ii)

We know that $\vec{F} = \nabla\phi$, therefore

$$\frac{\partial\phi}{\partial x} = 3x^2yz \;\; (1) \qquad\qquad \phi = x^3yz + f_1(y, z) \;\; (1a)$$

$$\frac{\partial\phi}{\partial y} = x^3z \;\; (2) \qquad\qquad \phi = x^3yz + f_2(x, z) \;\; (2a)$$

$$\frac{\partial\phi}{\partial z} = x^3y - 4z \;\; (3) \qquad\qquad \phi = x^3yz - 2z^2 + f_3(x, z) \;\; (3a)$$

Differentiate Eq. 1a with respect to y

$$\frac{\partial\phi}{\partial y} = x^3z + f_1'(y, z)$$

Set equal to Eq. 2

$$x^3z = x^3z + f_1'(y, z) \quad \Rightarrow \quad f_1'(y, z) = 0$$

Integrate $f_1'(y, z)$ with respect to y

$$f_1(y,z) = f_4(z)$$

Substitute into Eq. 1a

$$\phi = x^3yz + f_4(z) \qquad (4)$$

differentiate with respect to z

$$\frac{d\phi}{dz} = x^3y + f_4'(z)$$

Set equal to Eq. 3

$$x^3y + f_4'(z) = x^3y - 4z \;\;\Rightarrow\;\; f_4'(z) = -4z$$

Integration with respect to z gives

$$f_4(z) = -2z^2 + C$$

Substitute into Eq. 4

$$\phi = x^3yz - 2z^2 + C \quad \Leftarrow$$

Part iii)

The work done can be obtained from the scalar potential where $A = (1,2,7)$ and $B = (3,6,3)$

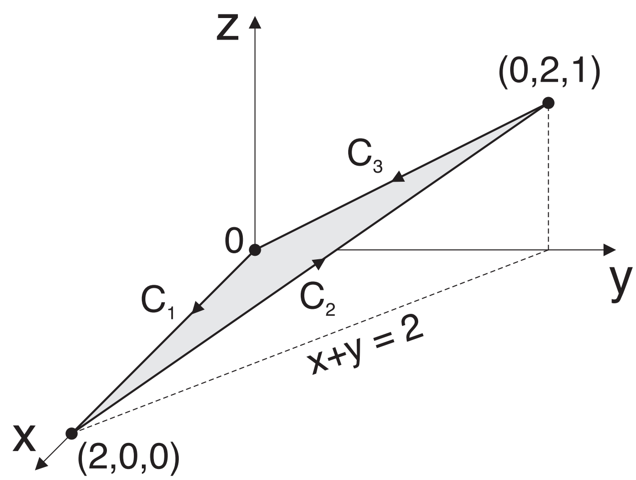
$$\begin{aligned} W = \phi_B - \phi_A &= [(3)^3(6)(3) - 2(3)^2 + C] - [(1)^3(2)(7) - 2(7)^2 + C] \\ &= 468 - (-84) = 552\,Joules \Leftarrow \end{aligned}$$

Question 5 (15 marks)

For the flow field

$$\vec{F} = \hat{i}(y^2) + \hat{j}(xy) - \hat{k}(2xz)$$

clearly show that Stokes' Theorem is valid for the triangular plane bounded by the sides C_1, C_2 and C_3 directed as shown in the figure below. (Show that LHS = RHS in Stokes' Theorem)



Stokes' theorem is given as

$$\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{LHS}} = \underbrace{\int\int_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS}_{\text{RHS}}$$

LHS

The three line integrals from C_1, C_2 and C_3 will be calculated individually and then added together. The integrand, $\vec{F} \cdot d\vec{r}$ is the same for all three integrals and is given as

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= [\hat{i}(y^2) + \hat{j}(xy) - \hat{k}(2xz)] \cdot [\hat{i}dx + \hat{j}dy + \hat{k}dz] \\ &= y^2dx + xydy - 2xzdz\end{aligned}$$

On C_1: $y = 0, \; z = 0, \; 0 \leq x \leq 2$

$$\vec{F} \cdot d\vec{r} = 0 \quad \Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = 0$$

On C_2: represent the line segment between $(2,0,0)$ and $(0,2,1)$ parametrically

$$\begin{aligned} x &= 2-2t \ , \ \ dx = -2dt \\ y &= 2t \ , \ \ dy = 2dt \\ z &= t \ , \ \ dz = dt \end{aligned}$$

for $0 \leq t \leq 1$. Therefore

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= 4t^2(-2dt) + (4t-4t^2)(2dt) - (4t-4t^2)dt \\ &= (4t - 12t^2)\ dt \end{aligned}$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{t=0}^{1} (4t - 12t^2)\ dt = -2$$

On C_3: $x = 0, \ \ z = y/2, \ \ 0 \leq y \leq 2$

$$\vec{F} \cdot d\vec{r} = 0 \quad \Rightarrow \int_{C_3} \vec{F} \cdot d\vec{r} = 0$$

Therefore summing over all three line segments gives

$$\oint_C \vec{F} \cdot d\vec{r} = 0 - 2 + 0 = -2 \Leftarrow$$

RHS

Given the counter clockwise rotation along the path C, the normal vector should point upward in the $+$ve $\hat{k}$ direction and to the left in the $-$ve $\hat{\jmath}$ direction.

The vector normal to the plane is perpendicular to the two vectors, $2\hat{\imath}$ and $2\hat{\jmath} + \hat{k}$ and is given by the cross product

$$\vec{n} = 2\hat{\imath} \ \times \ 2\hat{\jmath} + \hat{k} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix} = -2\hat{\jmath} + 4\hat{k}$$

The unit vector is given as

$$\hat{n} = \frac{-2\hat{\jmath} + 4\hat{k}}{\sqrt{4+16}} = \frac{-\hat{\jmath} + 2\hat{k}}{\sqrt{5}}$$

The equation of the plane can be determined from the normal and a point on the plane, i.e. $(0,0,0)$

$$0(x-0) - 2(y-0) + 4(z-0) = 0 \Rightarrow z = y/2$$

Therefore dS is

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \frac{\sqrt{5}}{2}\, dx\, dy$$

and

$$\begin{aligned} \hat{n} dS &= \left(\frac{-\hat{j} + 2\hat{k}}{\sqrt{5}}\right)\left(\frac{\sqrt{5}}{2}\right) dx\, dy \\ &= \left(-\frac{1}{2}\hat{j} + \hat{k}\right)\, dx\, dy \end{aligned}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & xy & -2xz \end{vmatrix} = 2z\hat{j} - y\hat{k}$$

Therefore

$$\begin{aligned} (\nabla \times F) \cdot \hat{n}\, dS &= (2z\hat{j} - y\hat{k}) \cdot \left(-\frac{1}{2}\hat{j} + \hat{k}\right)\, dx\, dy \\ &= (-z - y)\, dx\, dy \\ &= -\frac{3}{2} y \quad \text{(evaluated on the plane where } z = y/2) \end{aligned}$$

The RHS side of Stokes' theorem is

$$\begin{aligned} \int\int_S (\nabla \times F) \cdot \hat{n}\, dS &= -\frac{3}{2}\int_{x=0}^{2}\int_{y=0}^{2-x} y\, dy\, dx \\ &= -\frac{3}{4}\int_{x=0}^{2} (2-x)^2\, dx \\ &= -2 \Leftarrow \end{aligned}$$

$$LHS = RHS$$