ME 201 ADVANCED CALCULUS

8 February 2006

Midterm Examination R. Culham

- This is a two-hour, closed-book examination, where you are permitted to use:
  - Schaum's - Mathematical Handbook of Formulas and Tables
  - one 8.5 in.× 11 in. crib sheet. (one side)
- There are 6 questions to be answered. Read the questions very carefully.
- Clearly state all assumptions.
- It is your responsibility to write clearly and legibly.
- reduce all equations to their simplest possible form.

Question 1 (11 marks)

For the curve given by

$$\vec{R}(t) = \hat{i}(2\ln t) \; - \; \hat{j}\left(\frac{1}{t} + t\right) \; - \; \hat{k}$$

find:

i) velocity and acceleration vectors
ii) the angle (**radians**) between the velocity and acceleration vectors at $t = 1$
iii) the unit tangent vector
iv) the principal unit normal vector
v) the length of the curve over the interval $1 \leq t \leq 2$

Part i)

$$\vec{V} \;=\; \frac{d\vec{R}}{dt} = \hat{i}\left(\frac{2}{t}\right) \; + \; \hat{j}\left(\frac{1}{t^2} - 1\right) \Leftarrow$$

$$\vec{a} \;=\; \frac{d^2\vec{R}}{dt^2} = \hat{i}\left(-\frac{2}{t^2}\right) \; + \; \hat{j}\left(-\frac{2}{t^3}\right) \Leftarrow$$

Part ii) at $t = 1$

$$\vec{V}(1) \;=\; \hat{i}(2)$$

$$\vec{a}(1) \;=\; \hat{i}(-2) \; + \hat{j}(-2)$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{V}}{|\vec{a}||\vec{V}|} = -\frac{4}{(2)(2\sqrt{2})} = -\frac{1}{\sqrt{2}}$$

$$\theta = \frac{3\pi}{4}\ rad \Leftarrow$$

Part iii)

$$\hat{T} = \frac{\vec{V}}{|\vec{V}|}$$

$$|\vec{V}| = \sqrt{\left(\frac{2}{t}\right)^2 + \left(\frac{1}{t^2} - 1\right)^2}$$

$$= \sqrt{\frac{4t^2 + 1 - 2t^2 + t^4}{t^4}} = \frac{t^2+1}{t^2}$$

Therefore

$$\hat{T} = \hat{i}\left(\frac{2t}{1+t^2}\right) + \hat{j}\left(\frac{1-t^2}{1+t^2}\right) \Leftarrow$$

Part iv)

Noting that $\hat{T}\cdot\hat{N} = 0$ and that with 2D vectors we can obtain the normal by interchanging the coefficients of the unit tangent and taking the negative of the x coefficient

$$\hat{N} = -\hat{i}\left(\frac{1-t^2}{1+t^2}\right) + \hat{j}\left(\frac{2t}{1+t^2}\right) \Leftarrow$$

Part iv)

The length of the curve is

$$L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \sqrt{\left(\frac{2}{t}\right)^2 + \left(\frac{1}{t^2} - 1\right)^2}\,dt$$

$$= \int_1^2 \frac{t^2+1}{t^2}\,dt$$

$$= t - \frac{1}{t}\Big|_1^2 = \frac{3}{2} \Leftarrow$$

Question 2 (7 marks)

Find the curvature of the path

$$\vec{R}(t) = \hat{i}\ (e^t \cos t)\ +\ \hat{j}\ (e^t \sin t)\ +\ \hat{k}\ (e^t)$$

when $t = 0$.

The velocity vector is given as

$$\vec{V}(t) = \frac{d\vec{R}(t)}{dt} = \hat{i}\ e^t(\cos t - \sin t)\ +\ \hat{j}\ e^t(\cos t + \sin t)\ +\ \hat{k}\ (e^t)$$

and the acceleration vector

$$\vec{a}(t) = \frac{d\vec{R}^2(t)}{dt^2} = -\hat{i}\ (2e^t \sin t)\ +\ \hat{j}\ (2e^t \cos t)\ +\ \hat{k}\ (e^t)$$

But when $t = 0$

$$\begin{aligned} \vec{V}(0) &= \hat{i}\ (1)\ +\ \hat{j}\ (1)\ +\ \hat{k}\ (1) \\ \vec{a}(0) &= \hat{j}\ (2)\ +\ \hat{k}\ (1) \end{aligned}$$

$$\vec{V}(0) \times \vec{a}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \hat{i}\ (-1) + \hat{j}\ (-1) + \hat{k}\ (2)$$

$$\left|\vec{V}(0) \times \vec{a}(0)\right| = \sqrt{(-1)^2 + (-1)^2 + (2)^2} = \sqrt{6}$$

and

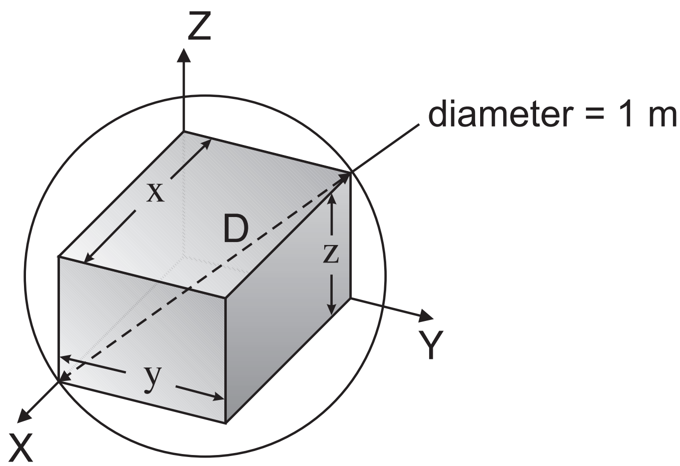
$$\left|\vec{V}(0)\right| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

The curvature is

$$\kappa = \frac{\left|\vec{V}(0) \times \vec{a}(0)\right|}{\left|\vec{V}(0)\right|^3} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3} \Leftarrow$$

Question 3 (11 marks)

Find the dimensions $(\boldsymbol{x, y, z})$ of a rectangular box of maximum volume that can be contained within a sphere with a diameter of $\boldsymbol{D = 1\ m}$, as shown below. Clearly demonstrate that this is the maximum volume. Note: the diagonal of the box will equal the diameter of the sphere.



Since we are attempting to maximize the volume of the box, let

$$\boldsymbol{V = x \cdot y \cdot z}$$

Since the diagonal of the box cannot exceed the diameter of the sphere, the problem is constrained such that

$$\boldsymbol{x^2 + y^2 + z^2 = 1}$$

We can then write $\boldsymbol{z}$ in terms of $\boldsymbol{x}$ and $\boldsymbol{y}$, thereby reducing the dimensional dependence of the problem by one variable

$$\boldsymbol{z^2 = 1 - x^2 - y^2}$$

For this problem, it is easier to minimize the function $\boldsymbol{f = V^2 = x^2y^2z^2}$ and if we minimize the square of the volume we also minimize the volume itself. Therefore

$$\boldsymbol{f = x^2y^2(1 - x^2 - y^2) = x^2y^2 - x^4y^2 - x^2y^4}$$

Determine the derivative of $\boldsymbol{f}$ with respect to the two independent variables and set them equal to zero to find the critical points

$$\boldsymbol{\frac{\partial f}{\partial x} = 2xy^2 - 4x^3y^2 - 2xy^4 = 2xy^2(1 - 2x^2 - y^2) = 0}$$

$$\boldsymbol{\frac{\partial f}{\partial y} = 2x^2y - 2x^4y - 4x^2y^3 = 2x^2y(1 - x^2 - 2y^2) = 0}$$

Since $\boldsymbol{x > 0}$ and $\boldsymbol{y > 0}$ (we expect a value of volume that is none zero), the terms in the brackets must go to zero to satisfy the above equations.

$$\boldsymbol{1 - 2x^2 - y^2 = 0} \tag{1}$$

$$\boldsymbol{1 - x^2 - 2y^2 = 0} \tag{2}$$

With 2 equations and 2 unknowns, we find

$$\boldsymbol{x = +\frac{1}{\sqrt{3}}} \qquad \text{and} \qquad \boldsymbol{y = +\frac{1}{\sqrt{3}}}$$

since $\boldsymbol{x, y > 0}$. Substituting back into the equation for the sphere, we find $\boldsymbol{z}$

$$\boldsymbol{z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - \frac{1}{3} - \frac{1}{3}} = +\frac{1}{\sqrt{3}}}$$

since $\boldsymbol{z > 0}$.

The critical point is $\boldsymbol{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$. Check for a relative maximum.

$$\boldsymbol{A = \left.\frac{\partial^2 f}{\partial x^2}\right|_{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)} = \frac{2}{3} - \frac{12}{9} - \frac{2}{9} = -0.8888}$$

$$\boldsymbol{B = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)} = \frac{4}{3} - \frac{8}{9} - \frac{8}{9} = -0.4444}$$

$$\boldsymbol{C = \left.\frac{\partial^2 f}{\partial y^2}\right|_{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)} = \frac{2}{3} - \frac{2}{9} - \frac{12}{9} = -0.8888}$$

$$\boldsymbol{D = B^2 - AC = (-0.4444)^2 - (-0.8888)((-0.8888) = -0.5925}$$

Since $\boldsymbol{D < 0}$ and $\boldsymbol{A < 0}$, we have a relative maximum and the maximum box dimensions are

$$\boldsymbol{x = \frac{1}{\sqrt{3}}\, m, \qquad y = \frac{1}{\sqrt{3}}\, m, \qquad z = \frac{1}{\sqrt{3}}\, m \Leftarrow}$$

and the maximum volume is

$$\boldsymbol{V = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = 0.19245\ m^3}$$

<u>Question 4</u> (9 marks)

Given the following two equations:

$$x^2 + y^2 - z^2 + 2xy = 1$$

$$x^3 + y^3 - 5y = 4$$

use the chain rule to find $\dfrac{dz}{dx}$.

The first equation can be written in terms of $z = f(x, y)$

$$z^2 = (x^2 + 2xy + y^2) - 1 = (x+y)^2 - 1$$

or

$$z = [(x+y)^2 - 1]^{1/2}$$

The second equation can be written in terms of $y = g(x)$

$$y^3 - 5y = 4 - x^3$$

Using the chain rule structure

```
   z
  / \
x    y
      \
       x
```

$$\Rightarrow \qquad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{1}{2}\Big[(x+y)^2 - 1\Big]^{-1/2} \cdot 2(x+y) \\ &= z^{-1/2} \cdot (x+y) = \frac{x+y}{z} \end{aligned}$$

Similarly

$$\begin{aligned} \frac{\partial z}{\partial y} &= \Big[(x+y)^2 - 1\Big]^{-1/2} \cdot (x+y) \\ &= z^{-1/2} \cdot (x+y) = \frac{x+y}{z} \end{aligned}$$

and finally

$$\begin{aligned}
3y^2\frac{dy}{dx} - 5\frac{dy}{dx} &= -3x^2 \\
\frac{dy}{dx}(5 - 3y^2) &= 3x^2 \\
\frac{dy}{dx} &= \frac{3x^2}{5 - 3y^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{dz}{dx} &= \left(\frac{x+y}{z}\right) + \left(\frac{x+y}{z}\right)\left(\frac{3x^2}{5-3y^2}\right) \\
&= \left(\frac{x+y}{z}\right)\left[1 + \frac{3x^2}{5-3y^2}\right] \Leftarrow
\end{aligned}$$

Question 5 (6 marks)

Find the distance between the planes:

$$\begin{aligned} \boldsymbol{x - 2y + 2z} &= \boldsymbol{5} \\ \boldsymbol{3x - 6y + 6z} &= \boldsymbol{30} \end{aligned}$$

The first thing to note is that the two planes are parallel, since the second equation can be written as

$$\boldsymbol{x - 2y + 2z = 10}$$

and we see that the unit normal vector for each plane is

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{\imath}} \boldsymbol{n_x} + \hat{\boldsymbol{\jmath}} \boldsymbol{n_y} + \hat{\boldsymbol{k}} \boldsymbol{n_z} = \hat{\boldsymbol{\imath}}(\boldsymbol{1}) + \hat{\boldsymbol{\jmath}}(\boldsymbol{-2}) + \hat{\boldsymbol{k}}(\boldsymbol{2})$$

Since both normal vectors are parallel, then both planes are parallel.

The distance between the planes can be determined by finding a point on one plane and then finding the distance between a point and a plane.

Select, $\boldsymbol{P(3, 1, 2)}$ as a point on the first plane. The second plane can be written as

$$\boldsymbol{Ax + By + Cz + D = 0} \qquad \rightarrow \qquad \boldsymbol{3x - 6y + 6z - 30 = 0}$$

The distance between $\boldsymbol{P}$ and the second plane is given as

$$\begin{aligned} \boldsymbol{d} &= \frac{|\boldsymbol{Ax_1 + By_1 + Cz_1 + D}|}{\sqrt{\boldsymbol{A^2 + B^2 + C^2}}} \\ &= \frac{|\boldsymbol{3(3) - 6(1) + 6(2) - 30}|}{\sqrt{\boldsymbol{3^2 + (-6)^2 + 6^2}}} \\ &= \frac{\boldsymbol{15}}{\boldsymbol{9}} = \frac{\boldsymbol{5}}{\boldsymbol{3}} \approx \boldsymbol{1.666} \Leftarrow \end{aligned}$$

Question 6 (11 marks)

Given the density function

$$f(x,y) = 6 - x^2y - 3xy^2$$

i) Find the maximum rate of density change at $(2,-1)$.
ii) Find the direction(s) of no density change at $(2,-1)$.
iii) Find the rate of density change at $(2,-1)$ away from the origin.
iv) Find the equation of the tangent to the graph $z = f(x,y)$ at the point $x = 2,\ y = -1$.

Part i)

$$\frac{\partial f}{\partial x} = -2xy - 3y^2$$

$$\frac{\partial f}{\partial y} = -x^2 - 6xy$$

$$\nabla f = \hat{\imath}\,(-2xy - 3y^2)\Big|_{x=2,y=-1} + \hat{\jmath}(-x^2 - 6xy)\Big|_{x=2,y=-1}$$

$$= \hat{\imath}(1) + \hat{\jmath}(8)$$

$$|\nabla f| = \sqrt{1^2 + 8^2} = \sqrt{65} \approx 8.062 \Leftarrow$$

Part ii)
The direction of no change in density at $(2,-1)$ is perpendicular or normal to the gradient vector

$$\vec{N} = \hat{\imath}(-8) + \hat{\jmath}(1)$$

or

$$\vec{N} = \hat{\imath}(8) + \hat{\jmath}(-1)$$

Part iii)
The vector at $(2,-1)$ away from the origin is given as

$$\vec{V} = \hat{\imath}(2-0) + \hat{\jmath}(-1-0) = \hat{\imath}(2) + \hat{\jmath}(-1)$$

The unit vector in this direction is given as

$$\hat{V} = \hat{\imath}\left(\frac{2}{\sqrt{5}}\right) + \hat{\jmath}\left(-\frac{1}{\sqrt{5}}\right)$$

The directional derivative is

$$\begin{aligned} D_V f = \nabla f \cdot \hat{V} &= \left[\hat{i}(1) + \hat{j}(8)\right] \cdot \left[\hat{i}\left(\frac{2}{\sqrt{5}}\right) + \hat{j}\left(-\frac{1}{\sqrt{5}}\right)\right] \\ &= -\frac{6}{\sqrt{5}} \approx -2.683 \end{aligned}$$

Part iv)
First write the equation of the curve as a function of (x, y, z)

$$F = f(x,y) - z = 6 - x^2 y - 3xy^2 - z = 0$$

At $(2, -1)$

$$z = 6 - x^2 y - 3xy^2 = 6 - (2)^2(-1) - 3(2)(-1)^2 = 4$$

$$\nabla F|_{x=2,y=-1} = \hat{i}(1) + \hat{j}(8) + \hat{k}(-1)$$

The equation of the plane is given as

$$\begin{aligned} N_x(x - x_1) + N_y(y - y_1) + N_z(z - z_1) &= 0 \\ 1 \cdot (x - 2) + 8 \cdot (y + 1) - 1 \cdot (z - 4) &= 0 \\ x + 8y - z &= -10 \Leftarrow \end{aligned}$$