University of Waterloo

ME201

ADVANCED CALCULUS
MIDTERM EXAMINATION

February 14, 2017 8:30 am - 10:30 am

Instructor: R. Culham

Name: ______________________________

Student ID Number: ______________________________

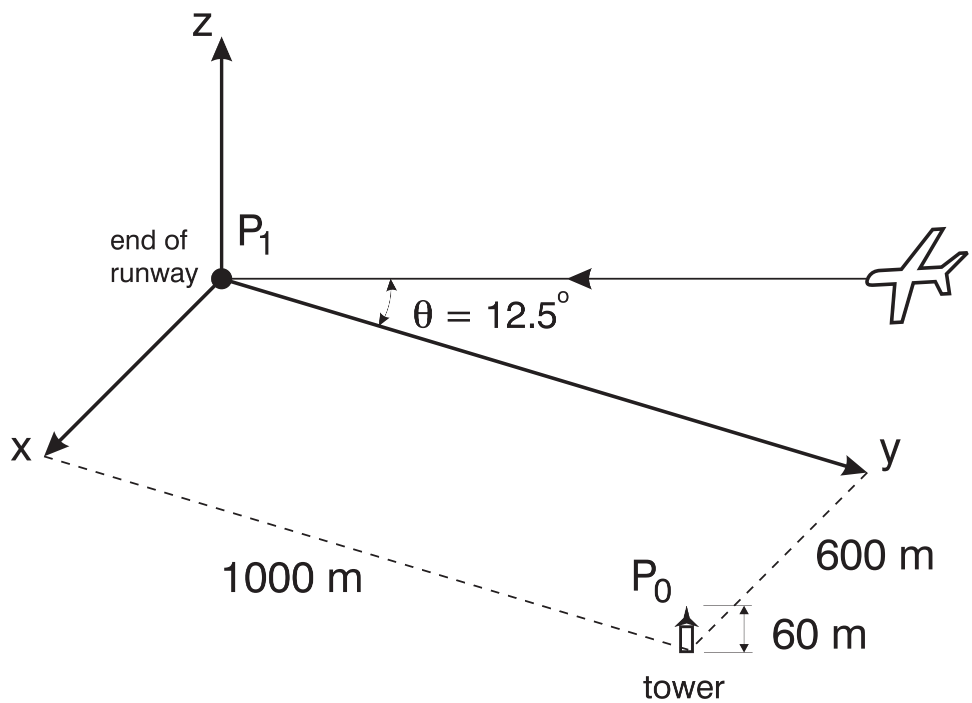
Instructions

1. This is a 2 hour, closed-book examination.
2. Permitted aids include:
   - one 8.5 in.× 11 in. crib sheet, (both sides)
   - Mathematical Handbook of Formulas and Tables, 4th ed., M.R. Spiegel, S. Lipschutz and J. Liu, Schaum's Outline Series, 2013.
   - calculator
3. Answer all questions in the space provided. If additional space is required, use the back of the pages or the blank pages included.
4. It is your responsibility to write clearly and legibly. Clearly state all assumptions. Part marks will be given for part answers, provided that your methodology is clear.

Question	**Marks**	**Grade**
1	8	
2	15	
3	8	
4	14	
5	10	
TOTAL	55	

Question 1 (8 marks)

An airplane approaches the end of a runway from the east with an angle of descent of $\mathbf{12.5^\circ}$. A tower that is $\mathbf{60}\ \boldsymbol{m}$ high is located $\mathbf{600}\ \boldsymbol{m}$ south and $\mathbf{1000}\ \boldsymbol{m}$ east of the end of the runway. How close does the plane come to a warning light that is at the top of the control tower?



The objective of this problem is to find the shortest distance between a position vector along the path of the planes descent and the point associated with the top of the control tower.

The top of the tower can be denoted by a point $\boldsymbol{P_0}(\mathbf{600, 1000, 60})$.

The position vector along the path of descent can be determined by finding two points along this path. We know one point is located at the point of touch down, $\boldsymbol{P_1}(\mathbf{0, 0, 0})$ and a second point, $\boldsymbol{P_2}$, can be found through basic trigonometry.

$$\begin{aligned} \boldsymbol{x_2} &= \mathbf{0} \\ \boldsymbol{y_2} &= \mathbf{1000} \\ \boldsymbol{z_2} &= \mathbf{1000 \cdot \tan\boldsymbol{\theta} = 1000 \cdot \tan 12.5^\circ = 221.7} \end{aligned}$$

Therefore we can write $\boldsymbol{P_2}(\mathbf{0, 1000, 221.7})$ and the position vector along the path of descent is

$$\vec{\boldsymbol{R}}_{\mathbf{12}} = \vec{\boldsymbol{R}}_{\mathbf{2}} - \vec{\boldsymbol{R}}_{\mathbf{1}} = \mathbf{1000}\hat{\boldsymbol{\jmath}} + \mathbf{221.7}\hat{\boldsymbol{k}}$$

The minimum distance between a point and a line is given as

$$
\begin{aligned}
\boldsymbol{d} &= \frac{|\vec{\boldsymbol{R}}_{\mathbf{10}} \times \vec{\boldsymbol{R}}_{\mathbf{12}}|}{|\vec{\boldsymbol{R}}_{\mathbf{12}}|} \\
&= \frac{|(\mathbf{600}\hat{\boldsymbol{i}} + \mathbf{1000}\hat{\boldsymbol{j}} + \mathbf{60}\hat{\boldsymbol{k}}) \times (\mathbf{1000}\hat{\boldsymbol{j}} + \mathbf{221.7}\hat{\boldsymbol{k}})|}{|(\mathbf{1000}\hat{\boldsymbol{j}} + \mathbf{221.7}\hat{\boldsymbol{k}})|} \\
&= \frac{\sqrt{(\mathbf{161,700})^2 + (\mathbf{-133,020})^2 + (\mathbf{600,000})^2}}{\sqrt{(\mathbf{1000})^2 + (\mathbf{221.7})^2}} \\
&= \mathbf{620.4}\ \boldsymbol{m} \Leftarrow
\end{aligned}
$$

<u>**Question 2**</u> (15 marks)

For the curve **C** formed by the intersection of $z = x^2 + 3y^2$ and $y = x - 1$:

(a) Express **C** in vector notation if the curve runs from $(0, -1, 3)$ to $(1, 0, 1)$

(b) Find the unit tangent vector to **C** at $\left(\frac{1}{2}, -\frac{1}{2}, 1\right)$

(c) Find the radius of curvature of **C** at $\left(\frac{1}{2}, -\frac{1}{2}, 1\right)$

(d) What is the unit tangent vector of **C** and the radius of curvature at $\left(\frac{1}{2}, -\frac{1}{2}, 1\right)$ if the curve, instead runs from $(1, 0, 1)$ to $(0, -1, 3)$

Part a)

First write the equation of the intersecting curve in parametric form. Set one variable equal to t, i.e. $x = t$ and solve for the other variables

$$\begin{aligned} x &= t \\ y &= t - 1 \\ z &= t^2 + 3(t-1)^2 = t^2 + 3t^2 - 6t + 3 = 4t^2 - 6t + 3 \end{aligned}$$

where the limits of t are

$$\begin{aligned} (0, -1, 3) &\longrightarrow t = 0 \\ (1, 0, 1) &\longrightarrow t = 1 \end{aligned}$$

The vector form of curve **C** can be written as

$$\vec{R}(t) = \hat{i}t + \hat{j}(t-1) + \hat{k}(4t^2 - 6t + 3) \quad : \quad 0 \le t \le 1 \qquad \Leftarrow \textbf{Part a)}$$

Part b)

The tangent vector can be written as

$$\vec{T} = \frac{d\vec{R}}{dt} = \hat{i} + \hat{j} + \hat{k}(8t - 6)$$

When evaluated at $\left(\frac{1}{2}, -\frac{1}{2}, 1\right)$ is equivalent to $t = \frac{1}{2}$ and

$$\vec{T}\Big|_{(1/2,-1/2,1)} = \hat{i} + \hat{j} + \hat{k}(-2)$$

and the unit tangent vector is

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{1^2 + 1^2 + (-2)^2}} = \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} - \frac{2}{\sqrt{6}}\hat{k} \qquad \Leftarrow \text{Part b)}$$

Part c)

The radius of curvature is given as $\rho = 1/\kappa$ where

$$\kappa = \frac{|R' \times R''|}{|R'|^3}$$

$$R' = \frac{d\vec{R}}{dt} = \hat{i} + \hat{j} - 2\hat{k}$$

$$R'' = \frac{d^2\vec{R}}{dt^2} = 8\hat{k}$$

$$R' \times R'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 0 & 0 & 8 \end{vmatrix} = 8\hat{i} - 8\hat{j} + 0\hat{k}$$

and

$$|R' \times R''| = \sqrt{8^2 + 8^2 + 0^2} = 8\sqrt{2}$$

$$|R'|^3 = (\sqrt{1^2 + 1^2 + (-2)^2})^3 = 6\sqrt{6}$$

and finally

$$\kappa = \frac{|R' \times R''|}{|R'|^3} = \frac{8\sqrt{2}}{6\sqrt{6}} = \frac{4}{3\sqrt{3}} = \frac{4\sqrt{3}}{9} = 0.7698$$

$$\rho = \frac{1}{\kappa} = \frac{3\sqrt{3}}{4} = 1.299 \qquad \Leftarrow \text{Part c)}$$

Part d)

The unit tangent vector points in the opposite direction of Part b)

$$\hat{T} = -\frac{1}{\sqrt{6}}\hat{i} - \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$$

The radius of curvature is unchanged at $\rho = 1.299$ $\Leftarrow$ **Part d)**

Question 3 (8 marks)

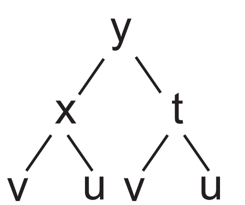
Use the chain rule to show that the partial differential equation

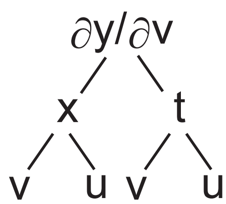
$$\frac{\partial^2 y}{\partial u\, \partial v} = 0$$

can be transformed into the one dimensional wave equation i.e. a vibrating string, which takes the following form

$$\frac{\partial^2 y}{\partial t^2} = 4\, \frac{\partial^2 y}{\partial x^2}$$

where the y is the amplitude of vibration, x is the position variable along the string given as $x = \frac{1}{2}(v+u)$ and t is the time given as $t = \frac{1}{4}(v-u)$.





$$\begin{aligned} \frac{\partial y}{\partial v} &= \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial y}{\partial t} \cdot \frac{\partial t}{\partial v} \\ &= \frac{\partial y}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial y}{\partial t}\left(\frac{1}{4}\right) \end{aligned}$$

since

$$\frac{\partial x}{\partial v} = \frac{1}{2} \qquad \text{and} \qquad \frac{\partial t}{\partial v} = \frac{1}{4}$$

$$\begin{aligned} \frac{\partial^2 y}{\partial u \partial v} = 0 = \frac{\partial\left(\frac{\partial y}{\partial v}\right)}{\partial u} &= \frac{\partial y}{\partial x}\left(\frac{\partial y}{\partial v}\right)\frac{\partial x}{\partial u} + \frac{\partial y}{\partial t}\left(\frac{\partial y}{\partial v}\right)\frac{\partial t}{\partial u} \\ &= \left[\frac{1}{2}\frac{\partial^2 y}{\partial x^2} + \frac{1}{4}\frac{\partial^2 y}{\partial x \partial t}\right] \cdot \left(\frac{1}{2}\right) \\ &\quad + \left[\frac{1}{2}\frac{\partial^2 y}{\partial t \partial x} + \frac{1}{4}\frac{\partial^2 y}{\partial t^2}\right] \cdot \left(-\frac{1}{4}\right) \\ 0 &= \frac{1}{4}\frac{\partial^2 y}{\partial x^2} + \frac{1}{8}\frac{\partial^2 y}{\partial x \partial t} - \frac{1}{8}\frac{\partial^2 y}{\partial t \partial x} - \frac{1}{16}\frac{\partial^2 y}{\partial t^2} \end{aligned}$$

Rearranging we obtain

$$\frac{\partial^2 y}{\partial t^2} = 4\, \frac{\partial^2 y}{\partial x^2} \quad \Leftarrow \boldsymbol{QED}$$

Question 4 (14 marks)

The local air temperature field $T(x, y, z)$ of the air in a room is given by

$$T(x, y, z) = 10 + x^2 + y^2 + z \qquad (°C)$$

where the origin of the co-ordinate system is one of the lower corners of the room. An insect is situated at $(1, 1, 1)$.

a) In which direction should the insect fly in order to warm up most rapidly? Specify the direction by a unit vector.
b) If the insect flies in this direction, at what rate will the air temperature seen by the insect increase? This increase should be specified in $°C/m$.
c) In which direction should the insect fly in order to remain at a constant temperature?
d) Suppose instead the insect flew in the direction of vector $\vec{V} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$. At what rate, in $°C/m$, will the temperature seen by the insect change then?

Part a)

The temperature is given by

$$T(x, y, z) = 10 + x^2 + y^2 + z \qquad (°C)$$

and the insect starts at $(1, 1, 1)$.
To warm up most rapidly, the insect should fly in the direction of the gradient, given by

$$\begin{aligned} \nabla T &= \frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k} \\ &= 2x\,\hat{\imath} + 2y\,\hat{\jmath} + \hat{k} \end{aligned}$$

Evaluating the gradient at $(1, 1, 1)$ gives

$$\nabla T|_{(1,1,1)} = 2\,\hat{\imath} + 2\,\hat{\jmath} + \hat{k}$$

As a unit vector this is

$$\frac{2}{3}\,\hat{\imath} + \frac{2}{3}\,\hat{\jmath} + \frac{1}{3}\,\hat{k} \qquad \Leftarrow \textbf{Part a)}$$

Part b)

The rate of increase in the air temperature in this direction is given as

$$|\nabla T| = \sqrt{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2} = \sqrt{2^2 + 2^2 + 1^2} = 3\ °C/m \Leftarrow \textbf{Part b)}$$

Part c)

Because the temperature is defined in a 3D space, we need to find the tangent plane at point $(1,1,1)$. This plane contains a full complment of vectors that are tangent to the gradient and hence are tangent to the level surface at $(1,1,1)$. We know the gradient is given by

$$\nabla T = A\hat{i} + B\hat{j} + C\hat{k}$$

where

$$\begin{aligned} A &= \left.\frac{\partial T}{\partial x}\right|_{(1,1,1)} = 2 \\ B &= \left.\frac{\partial T}{\partial y}\right|_{(1,1,1)} = 2 \\ C &= \left.\frac{\partial T}{\partial x}\right|_{(1,1,1)} = 1 \end{aligned}$$

The tangent plane is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$\begin{aligned} 2(x-1) + 2(y-1) + (z-1) &= 0 \\ 2x - 2 + 2y - 2 + z - 1 &= 0 \\ 2x + 2y + z &= 5 \qquad \Leftarrow \textbf{Part c)} \end{aligned}$$

Aside: as a check, see if a vector in this plane dotted with the gradient is equal to zero (a necessary condition for perpendicular vectors)
Two points in the tangent plane are $(1,1,1)$ and $(0,0,5)$. The vector between these two points is

$$1\hat{i} + 1\hat{j} - 4\hat{k}$$

$$(1\hat{i} + 1\hat{j} - 4\hat{k}) \cdot (2\hat{i} + 2\hat{j} + \hat{k}) = 0$$

Therefore the vectors are perpendicular.

Part d)

If the insect flew in the direction of the vector, $\vec{\boldsymbol{V}} = \hat{\boldsymbol{\imath}} - \boldsymbol{3}\hat{\boldsymbol{\jmath}} - \boldsymbol{5}\hat{\boldsymbol{k}}$, then the temperature would increase at a rate

$$\boldsymbol{D_v T = \nabla T \cdot \hat{V}}$$

where

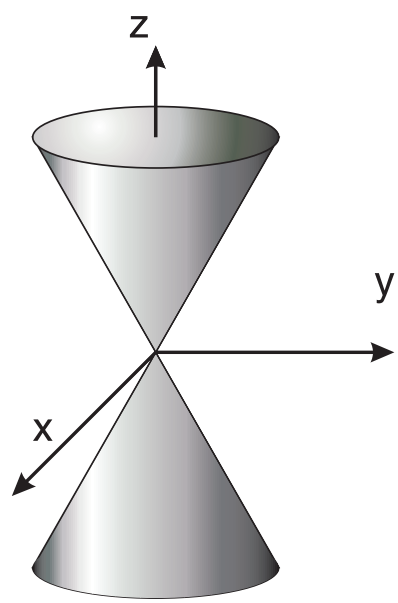
$$\hat{\boldsymbol{V}} = \frac{\vec{\boldsymbol{V}}}{\left|\vec{\boldsymbol{V}}\right|} = \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{35}}}\hat{\boldsymbol{\imath}} - \frac{\boldsymbol{3}}{\sqrt{\boldsymbol{35}}}\hat{\boldsymbol{\jmath}} - \frac{\boldsymbol{5}}{\sqrt{\boldsymbol{35}}}\hat{\boldsymbol{k}}$$

and

$$\begin{aligned}
\boldsymbol{D_v T} &= (\boldsymbol{2}\ \hat{\boldsymbol{\imath}} + \boldsymbol{2}\ \hat{\boldsymbol{\jmath}} + \boldsymbol{1}\ \hat{\boldsymbol{k}}) \cdot \left(\frac{\boldsymbol{1}}{\sqrt{\boldsymbol{35}}}\hat{\boldsymbol{\imath}} - \frac{\boldsymbol{3}}{\sqrt{\boldsymbol{35}}}\hat{\boldsymbol{\jmath}} - \frac{\boldsymbol{5}}{\sqrt{\boldsymbol{35}}}\hat{\boldsymbol{k}}\right) \\
&= \frac{\boldsymbol{2}}{\sqrt{\boldsymbol{35}}} - \frac{\boldsymbol{6}}{\sqrt{\boldsymbol{35}}} - \frac{\boldsymbol{5}}{\sqrt{\boldsymbol{35}}} \\
&= -\frac{\boldsymbol{9}}{\sqrt{\boldsymbol{35}}}\ {}^\circ \boldsymbol{C/m} \qquad \Leftarrow \textbf{Part d)}
\end{aligned}$$

Question 5 (10 marks)

Find the point(s) on the surface of the elliptic cone $3x^2 + 2y^2 = z^2$ that are closest to the point $(1, 1, 0)$. Hint: Consider locating a sphere of radius, r, at point $(1, 1, 0)$.



This is a problem requiring a minimization procedure where we need to find a sphere with the minimum radius r that is centered about the point $(1, 1, 0)$. The equation of the sphere is given by

$$f = r^2 = (x - 1)^2 + (y - 1)^2 + z^2$$

The minimum distance or radius of the sphere can be obtained by finding the intersection of these two curves and then minimizing the function f and in turn r. Combining the equation of the cone and the sphere gives

$$\begin{aligned} f = r^2 &= (x-1)^2 + (y-1)^2 + 3x^2 + 2y^2 \\ &= x^2 - 2x + 1 + y^2 - 2y + 1 + 3x^2 + 2y^2 \\ &= 4x^2 - 2x + 3y^2 - 2y + 2 \end{aligned}$$

Taking the derivative first with respect to x and then with respect to y and setting these derivatives equal to 0 allows us to obtain the critical point

$$\begin{aligned} \frac{df}{dx} &= 8x - 2 = 0 \\ \frac{df}{dy} &= 6y - 2 = 0 \end{aligned}$$

The simultaneous solution to these equations is

$$\begin{aligned} \boldsymbol{x} &= \mathbf{1/4} \\ \boldsymbol{y} &= \mathbf{1/3} \end{aligned}$$

Which leads to

$$\boldsymbol{z^2 = 3x^2 + 2y^2 = \frac{27+32}{144} = \frac{59}{144}}$$

or

$$\boldsymbol{z = \pm\frac{\sqrt{59}}{12}}$$

The points on the surface of the elliptic cone closest to $\mathbf{(1,1,0)}$ are $\boldsymbol{\left(\frac{1}{4}, \frac{1}{3}, \pm\frac{\sqrt{59}}{12}\right)}$ $\Leftarrow$