University of Waterloo

ME201

ADVANCED CALCULUS
MIDTERM EXAMINATION

February 13, 2018 8:30 am - 10:30 am

Instructor: R. Culham

Name: ______________________________

Student ID Number: ______________________________

Instructions

1. This is a 2 hour, closed-book examination.
2. Permitted aids include:
   - one 8.5 in.× 11 in. crib sheet, (both sides)
   - Mathematical Handbook of Formulas and Tables, M.R. Spiegel, S. Lipschutz and J. Liu, Schaum's Outline Series.
   - calculator
3. Answer all questions in the space provided. If additional space is required, use the back of the pages or the blank pages included.
4. It is your responsibility to write clearly and legibly. Clearly state all assumptions. Part marks will be given for part answers, provided that your methodology is clear.

Question	**Marks**	**Grade**
1	9	
2	9	
3	12	
4	15	
5	15	
TOTAL	60	

<u>**Question 1**</u> (9 marks)

The intersection of a sphere, $x^2 + y^2 + z^2 = 4a^2$ and a cylinder, $x^2 + (y - a)^2 = a^2$ in the positive x, y, z quadrant (i.e. $x \geq 0, y \geq 0, z \geq 0$) produces a curve defined by the parametric equations

$$\begin{aligned} x &= a \sin 2t \\ y &= a + a \cos 2t \\ z &= 2a \sin t \end{aligned}$$

a) what is the range of t for the curve of intersection
b) calculate the unit tanget vector along the curve in the direction of increasing t at $t = 0$
c) find the principal unit normal vector at $t = 0$. Calculations are not necessary if you can reasonably explain its value.
d) Set up the integral from which you could find the arc length of the curve over the range of t obtained in part a). Do not evaluate the integral but reduce your answer to its simplest form.

Part a)

At $z = 0 \;\Rightarrow\; \boxed{t = 0}$ and at $y = 0 \;\; a \cos 2t = -a \;\Rightarrow\; \boxed{t = \pi/2}$

Part b)

The position vector defining the line of intersection is given by

$$\vec{R}(t) = \hat{\imath}(a \sin 2t) + \hat{\jmath}(a + a \cos 2t) + \hat{k}(2a \sin t)$$

The unit tangent is given by

$$\hat{T}(t) = \frac{d\vec{R}(t)/dt}{\left|d\vec{R}(t)/dt\right|} = \frac{\hat{\imath}(2a \cos 2t) + \hat{\jmath}(-2a \sin 2t) + \hat{k}(2a \cos t)}{\sqrt{4a^2 \cos^2 2t + 4a^2 \sin^2 2t + 4a^2 \cos^2 t}}$$

$$= \hat{\imath}\left(\frac{\cos 2t}{\sqrt{1 + \cos^2 t}}\right) - \hat{\jmath}\left(\frac{\sin 2t}{\sqrt{1 + \cos^2 t}}\right) + \hat{k}\left(\frac{\cos t}{\sqrt{1 + \cos^2 t}}\right)$$

At $t = 0$

$$\hat{T}(t) = \hat{\imath}\left(\frac{1}{\sqrt{2}}\right) + \hat{k}\left(\frac{1}{\sqrt{2}}\right) \Leftarrow$$

Part c)

$$\hat{N} = \hat{\jmath} \Leftarrow$$

Reasons:

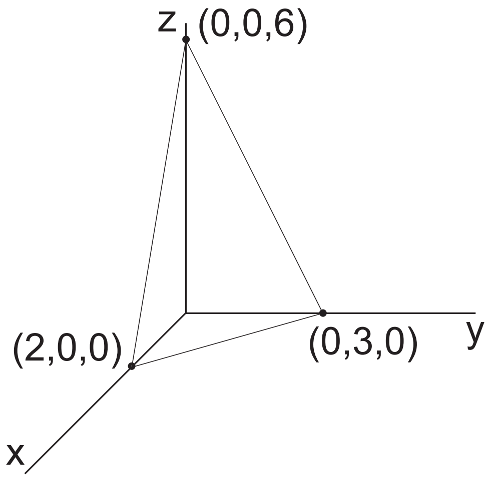
1. from the general equation for $\hat{\boldsymbol{T}}$ we can see that the derivative of the tangent evaluated at $\boldsymbol{t} = \mathbf{0}$ will result in the terms associated with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{k}}$ going to zero, as the **sin** terms go to zero at $\boldsymbol{t} = \mathbf{0}$. Therefore the direction is only a function of $\hat{\boldsymbol{\jmath}}$. Since the time variable is increasing in the positive$\boldsymbol{y}-$ direction, the principal unit normal must be $\hat{\boldsymbol{N}} = \hat{\boldsymbol{\jmath}}$

2. at $\boldsymbol{t} = \mathbf{0}$ the normal to the curve of intersection is identical to the normal to both the cylinder and the sphere. At this point, i.e. $\boldsymbol{t} = \mathbf{0}$ this is clearly in the direction of the origin or $\hat{\boldsymbol{N}} = \hat{\boldsymbol{\jmath}}$

Part d)
The length of the curve is given by

$$
\begin{aligned}
\boldsymbol{L} &= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^{\pi/2} \sqrt{(2a\cos 2t)^2 + (-2a\sin 2t)^2 + (2a\cos t)^2}\, dt \\
&= 2a\int_0^{\pi/2} \sqrt{1 + cos^2 t}\, dt \Leftarrow
\end{aligned}
$$

Question 2 (9 marks)



Given a line formed by the parametric equations

$$\boldsymbol{x} = \boldsymbol{6 - 4t}$$

$$\boldsymbol{y} = \boldsymbol{1 + 5t}$$

$$\boldsymbol{z} = \boldsymbol{2 + 2t}$$

and the triangular plane given as $\boldsymbol{3x+2y+z = 6}$ (as illustrated)

a) clearly show that the line and the plane are parallel

b) find the shortest distance between the line and the plane

Part a)

We can find the vector form of the line by taking any two points on the line and then find the vector between these two points.

$$@\ \boldsymbol{t = 0} \quad \Rightarrow \quad \boldsymbol{P_0(6, 1, 2)}$$

$$@\ \boldsymbol{t = 1} \quad \Rightarrow \quad \boldsymbol{P_1(2, 6, 4)}$$

The position vector is given as

$$\overrightarrow{\boldsymbol{P_0P_1}} = \boldsymbol{-4\hat{i} + 5\hat{j} + 2\hat{k}}$$

We know that the general equation of a plane is given as

$$\boldsymbol{Ax + By + Cz + D = 0}$$

and (A,B,C) are the components of the normal vector. Therefore in our case

$$\vec{\boldsymbol{n}} = \boldsymbol{3\hat{i} + 2\hat{j} + 1\hat{k}}$$

We know that if $\overrightarrow{\boldsymbol{P_0P_1}}$ and the normal to the plane, $\vec{\boldsymbol{n}}$ are perpendicular to one another then the dot product will be zero and the plane and the line will be parallel.

Taking the dot product of the position vector and the normal vector, we see

$$\begin{aligned}\overrightarrow{\boldsymbol{P_0P_1}} \cdot \vec{\boldsymbol{n}} &= (-4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) \cdot (3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 1\hat{\boldsymbol{k}}) \\ &= -12 + 10 + 2 \\ &= 0 \quad \Leftarrow\end{aligned}$$

Since the dot product is zero, the line and the normal are perpendicular and we have clearly shown that the line and the plane are parallel.

Part b)

We have two points on the line and the equation of the plane, therefore we can use the formulation for determining the distance between a plane and a point.

$$\begin{aligned}\textbf{shortest distance} &= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|3(6) + 2(1) + 1(2) - 6|}{\sqrt{9 + 4 + 1}} \\ &= \frac{16}{\sqrt{14}} \quad \Leftarrow\end{aligned}$$

Question 3 (12 marks)

A function $f(x,t)$ is governed by the partial differential equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + e^{-t}\sin x$$

The independent variable x is to be replaced by a new independent variable u, where

$$u = (x+t)^2$$

Use the chain rule to derive the differential equation corresponding to the one above which will govern the function $f(u,t)$.

Your final equation should contain only the variables u and t and partial derivatives of f with respect to u and t; that is, all terms in x should be eliminated.

The function $f(x,t)$ obeys

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + e^{-t}\sin x$$

We can introduce the new independent variable

$$u = (x+t)^2 \quad \Rightarrow \quad \frac{\partial u}{\partial t} = 2(x+t)$$

or in terms of x

$$x = \sqrt{u} - t$$

Given that $f = f(u,t)$ and $u = f(x,t)$, the chain rule tree will look like

$$\begin{array}{ccccc} & & f & & \\ & \swarrow & & \searrow & \\ & u & & & t \\ \swarrow & & \searrow & & \\ x & & & t & \end{array}$$

As a result we can say

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial f}{\partial t}$$

Therefore

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\cdot 2(x+t) + \frac{\partial f}{\partial t} = 2\sqrt{u}\frac{\partial f}{\partial u} + \frac{\partial f}{\partial t}$$

From the tree structure we see

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = 2(x+t)\frac{\partial f}{\partial u} = 2\sqrt{u}\frac{\partial f}{\partial u}$$

and

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial x}\right)\frac{\partial u}{\partial x} \\
&= \left[\frac{\partial}{\partial u}\left(2\sqrt{u}\frac{\partial f}{\partial u}\right)\right] \cdot 2\sqrt{u} \\
&= 4u\frac{\partial^2 f}{\partial u^2} + 4\sqrt{u} \cdot \frac{1}{2\sqrt{u}}\frac{\partial f}{\partial u}
\end{aligned}$$

and finally

$$e^{-t}\sin x = e^{-t}\sin(\sqrt{u} - t)$$

Substituting back into our original equation gives

$$2\sqrt{u}\frac{\partial f}{\partial u} + \frac{\partial f}{\partial t} = 4u\frac{\partial^2 f}{\partial u^2} + 2\frac{\partial f}{\partial u} + e^{-t}\sin(\sqrt{u} - t)$$

Therefore equation that governs $f(u, t)$ is

$$\frac{\partial f}{\partial t} = 2(1 - \sqrt{u})\frac{\partial f}{\partial u} + 4u\frac{\partial^2 f}{\partial u^2} + e^{-t}\sin(\sqrt{u} - t) \Leftarrow$$

<u>**Question 4**</u> (15 marks)

a) Find the directional derivative of the function $f = z\tan^{-1}\dfrac{x}{y}$ at the point , $P_0(1,1,2)$ in the direction $A = -6\hat{i} + 2\hat{j} - 3\hat{k}$.

b) What is the maximum and minimum rate of change of f at P_0, and in what direction do these occur?

c) Briefly describe the meaning of the directional derivative. Using the information in part a), make a sketch that clearly illustrates the meaning of the directional derivative.

Part a)

From Schaum's Eq. 15.22, we can find the derivative for $\tan^{-1}$. Therefore

First we must find the derivative of f with respect to x, y and z,

$$\frac{\partial f}{\partial x} = z \cdot \frac{1}{1+(x/y)^2} \cdot \frac{1}{y} = \frac{zy}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = z \cdot \frac{1}{1+(x/y)^2} \cdot \frac{-x}{y^2} = \frac{-zx}{x^2+y^2}$$

$$\frac{\partial f}{\partial z} = \tan^{-1}\frac{x}{y}$$

Therefore

$$\nabla f|_{(P_0)} = \frac{\partial f(1,1,2)}{\partial x}\,\hat{i} + \frac{\partial f(1,1,2)}{\partial y}\,\hat{j} + \frac{\partial f(1,1,2)}{\partial z}\,\hat{k}$$

$$= (1)\hat{i} + (-1)\hat{j} + \left(\frac{\pi}{4}\right)\hat{k}$$

The unit vector in the A direction is given as

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{\sqrt{36+4+9}} = -\frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}$$

Therefore the directional derivative of f in the direction $\hat{A}$ at P_0 is

$$D_{\hat{A}}f|_{P_0} = \nabla f \cdot \hat{A}\Big|_{P_0} = \left[(1)\hat{i} + (-1)\hat{j} + \left(\frac{\pi}{4}\right)\hat{k}\right] \cdot \left[-\frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}\right]$$

$$= -\frac{6}{7} - \frac{2}{7} - \frac{3\pi}{28}$$

$$= -\frac{32 + 3\pi}{28} \approx -1.48 \quad \Leftarrow$$

Part b)

The direction of the maximum rate of change is in the direction of the gradient, ∇f.

$$\nabla f = (1)\hat{i} + (-1)\hat{j} + \left(\frac{\pi}{4}\right)\hat{k}$$

The value of the directional derivative in that direction is

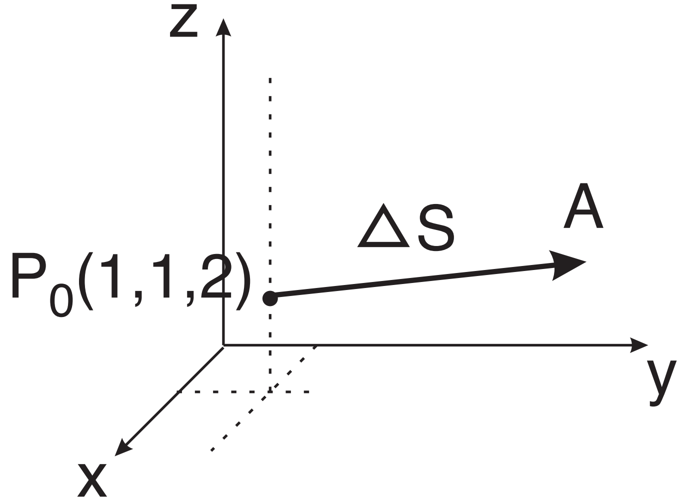
$$D_{max} f = |\nabla f| = \sqrt{1 + 1 + (\pi/4)^2} = \frac{1}{4}\sqrt{32 + \pi^2} \approx 1.62 \quad \Leftarrow$$

The minimum value of the directional derivative will be along the level surface and will have a value of zero.

$$D_{min} f = 0 \quad \Leftarrow$$

Part c)

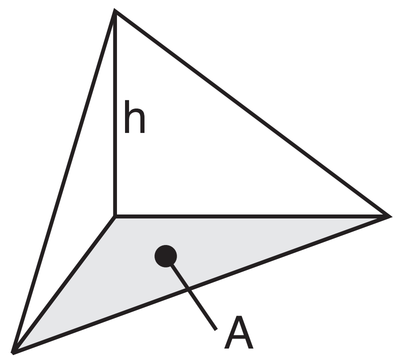
The directional derivative gives the rate at which the function $f(x, y, z)$ changes as it moves between level surfaces following the A-direction. In our case, for each unit travelled (ΔS) along the A-vector starting at the point P_0, the rate of change in the function f is -1.48 units.



<u>**Question 5**</u> (15 marks)

Find the equation of the plane which passes through the point $\boldsymbol{(2,1,4)}$ and cuts off the least volume from the first octant (i.e. $\boldsymbol{x \geq 0, y \geq 0, z \geq 0}$).

Note: The solid volume of the shape shown below can be calculated as **Volume** $\boldsymbol{= \frac{1}{3}Ah}$.



The general equation for a plane can be written as

$$\boldsymbol{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

But if we let $\boldsymbol{D = A/C}$ and $\boldsymbol{E = B/C}$ we can reduce the number of unknowns by $\boldsymbol{1}$ and the plane becomes

$$\boldsymbol{D(x - x_0) + E(y - y_0) + (z - z_0) = 0}$$

Since the plane has to pass through $\boldsymbol{(2,1,4)}$, the equation of the plane is

$$\boldsymbol{D(x - 2) + E(y - 1) + (z - 4) = 0}$$

or

$$\boldsymbol{Dx + Ey + z = 2D + E + 4}$$

Since we know the volume resides in the 1st octant, the intercepts along the $\boldsymbol{x, y, z}$-axes can be found as follows:

$$\text{x-intercept, let } \boldsymbol{y = 0,\ z = 0 \ \Rightarrow\ x_i = \frac{2D + E + 4}{D}}$$

$$\text{y-intercept, let } \boldsymbol{x = 0,\ z = 0 \ \Rightarrow\ y_i = \frac{2D + E + 4}{E}}$$

$$\text{z-intercept, let } \boldsymbol{x = 0,\ y = 0 \ \Rightarrow\ z_i = 2D + E + 4}$$

We need to minimize the volume, where $\boldsymbol{A = \frac{1}{2}x_i y_i}$ and $\boldsymbol{h = z_i}$, therefore

$$V = \frac{1}{6} x_i y_i z_i = \frac{(2D + E + 4)^3}{6DE}$$

where

$$\frac{\partial V}{\partial D} = 0 \qquad \text{and} \qquad \frac{\partial V}{\partial E} = 0$$

$$\frac{\partial V}{\partial D} = \frac{(3)(2D + E + 4)^2(2)(6DE) - (2D + E + 4)^3(6E)}{36D^2E^2} = 0 \tag{1}$$

$$\frac{\partial V}{\partial E} = \frac{(3)(2D + E + 4)^2(1)(6DE) - (2D + E + 4)^3(6D)}{36D^2E^2} = 0 \tag{2}$$

We see that $D \neq E \neq 0$, so we need to find D and E that force the numerators to $\mathbf{0}$.

From Eq. (1)

$$\begin{aligned} 36DE(2D + E + 4)^2 &= 6E(2D + E + 4)^3 \\ 4D &= E + 4 \end{aligned}$$

and from Eq. (2)

$$\begin{aligned} 18DE(2D + E + 4)^2 &= 6D(2D + E + 4)^3 \\ 2E &= 2D + 4 \\ E &= D + 4 \end{aligned}$$

Solving these two equations simultaneously gives

$$D = 2 \qquad \text{and} \qquad E = 4$$

and the equation of the plane that minimizes the volume is

$$2x + 4y + z = 12 \;\; \Leftarrow$$