University of Waterloo

ME201

ADVANCED CALCULUS
MIDTERM EXAMINATION

February 13, 2018 8:30 am - 10:30 am

Instructor: R. Culham

Name: ____________________

Student ID Number: ____________________

Instructions

1. This is a 2 hour, closed-book examination.
2. Permitted aids include:
   - one 8.5 in.$\times$ 11 in. crib sheet, (both sides)
   - Mathematical Handbook of Formulas and Tables, M.R. Spiegel, S. Lipschutz and J. Liu, Schaum's Outline Series.
   - calculator
3. Answer all questions in the space provided. If additional space is required, use the back of the pages or the blank pages included.
4. It is your responsibility to write clearly and legibly. Clearly state all assumptions. Part marks will be given for part answers, provided that your methodology is clear.

Question	**Marks**	**Grade**
1	9	
2	9	
3	12	
4	15	
5	15	
TOTAL	60	

Question 1 (9 marks)

The intersection of a sphere, $x^2 + y^2 + z^2 = 4a^2$ and a cylinder, $x^2 + (y - a)^2 = a^2$ in the positive x, y, z quadrant (i.e. $x \geq 0, y \geq 0, z \geq 0$) produces a curve defined by the parametric equations

$$x = a \sin 2t$$

$$y = a + a \cos 2t$$

$$z = 2a \sin t$$

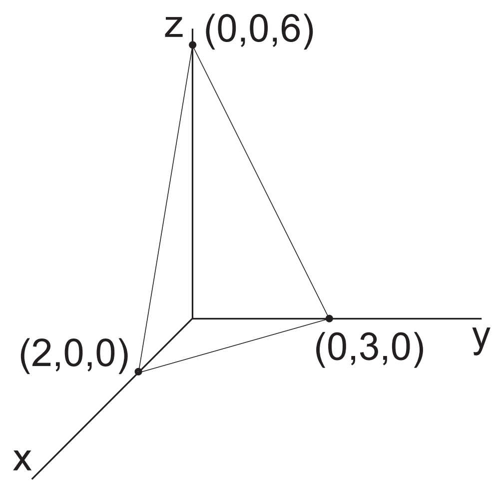
a) what is the range of t for the curve of intersection

b) calculate the unit tanget vector along the curve in the direction of increasing t at $t = 0$

c) find the principal unit normal vector at $t = 0$. Calculations are not necessary if you can reasonably explain its value.

d) Set up the integral from which you could find the arc length of the curve over the range of t obtained in part a). Do not evaluate the integral but reduce your answer to its simplest form.

Question 2 (9 marks)



Given a line formed by the parametric equations

$$\begin{aligned} \boldsymbol{x} &= \mathbf{6 - 4}\boldsymbol{t} \\ \boldsymbol{y} &= \mathbf{1 + 5}\boldsymbol{t} \\ \boldsymbol{z} &= \mathbf{2 + 2}\boldsymbol{t} \end{aligned}$$

and the triangular plane given as $\mathbf{3}\boldsymbol{x}\mathbf{+2}\boldsymbol{y}\mathbf{+}\boldsymbol{z} = \mathbf{6}$ (as illustrated)

a) clearly show that the line and the plane are parallel

b) find the shortest distance between the line and the plane

Question 3 (12 marks)

A function $f(x, t)$ is governed by the partial differential equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + e^{-t} \sin x$$

The independent variable x is to be replaced by a new independent variable u, where

$$u = (x + t)^2$$

Use the chain rule to derive the differential equation corresponding to the one above which will govern the function $f(u, t)$.

Your final equation should contain only the variables u and t and partial derivatives of f with respect to u and t; that is, all terms in x should be eliminated.

Question 4 (15 marks)

a) Find the directional derivative of the function $\boldsymbol{f} = \boldsymbol{z} \tan^{-1} \dfrac{\boldsymbol{x}}{\boldsymbol{y}}$ at the point , $\boldsymbol{P_0(1, 1, 2)}$ in the direction $\boldsymbol{A} = -6\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$.

b) What is the maximum and minimum rate of change of $\boldsymbol{f}$ at $\boldsymbol{P_0}$, and in what direction do these occur?

c) Briefly describe the meaning of the directional derivative. Using the information in part a), make a sketch that clearly illustrates the meaning of the directional derivative.

Question 5 (15 marks)

Find the equation of the plane which passes through the point $\mathbf{(2, 1, 4)}$ and cuts off the least volume from the first octant (i.e. $\boldsymbol{x \geq 0, y \geq 0, z \geq 0}$).

Note: The solid volume of the shape shown below can be calculated as $\textbf{Volume} = \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{Ah}$.