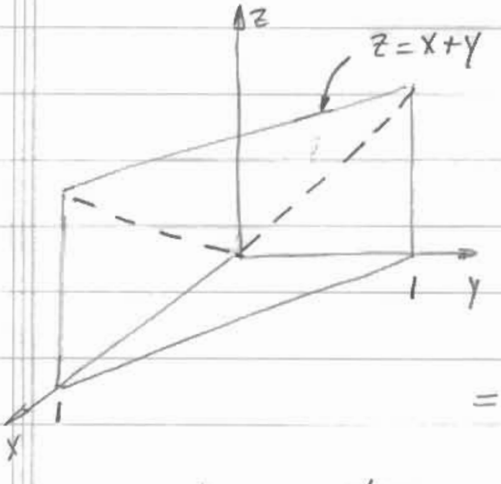
ASSIGNMENT #11 SOLUTIONS

1a) $\iint_S (x^2+y^2)z\,dS$ $\qquad S:\ z = x+y \qquad x=0 \quad x+y=1 \quad y=0$



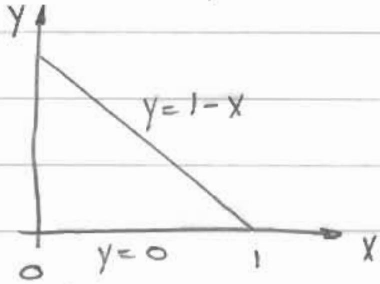
$\Rightarrow$ project onto xy plane

$$\iint_S (x^2+y^2)z\,dS$$

$$= \iint_{R_{xy}} (x^2+y^2)(x+y)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \sqrt{3}\iint_{R_{xy}} x^3+x^2y+xy^2+y^3\,dA$$

examine xy plane



$$= \sqrt{3}\int_0^1\int_0^{1-x} x^3+x^2y+xy^2+y^3\,dy\,dx$$

$$= \sqrt{3}\int_0^1 x^3y+\frac{x^2y^2}{2}+\frac{xy^3}{3}+\frac{y^4}{4}\Big|_0^{1-x}\,dx$$

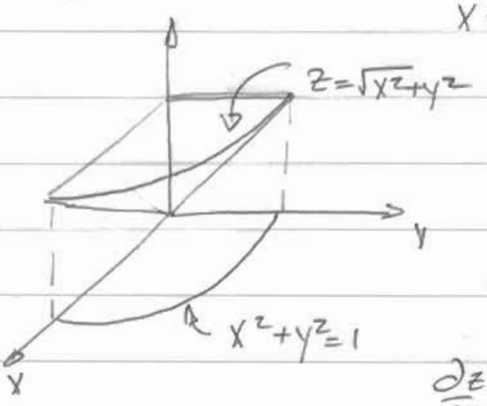
$$= \sqrt{3}\int_0^1 x^3-x^4+\frac{x^2}{2}(1-2x+x^2)+\frac{x}{3}(1-3x+3x^2-x^3)+\frac{1}{4}(1-4x+6x^2-4x^3+x^4)\,dx$$

$$= \sqrt{3}\int_0^1 -\frac{7}{12}x^4+x^2-\frac{2}{3}x+\frac{1}{4}\,dx = \sqrt{3}\left[-\frac{7}{12}\cdot\frac{1}{5}+\frac{1}{3}-\frac{2}{3}\cdot\frac{1}{2}+\frac{1}{4}\right]$$

$$= \frac{2\sqrt{3}}{15}$$

b) $\iint_S xy\,dS$ $\qquad S:\ z=\sqrt{x^2+y^2}$ ← cone

$x^2+y^2=1$ ← circular cylinder



∴ project on xy plane

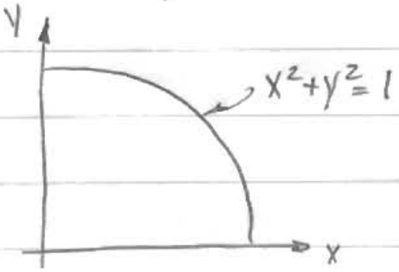
$$= \iint_{R_{xy}} xy\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$= \iint_{R_{xy}} xy\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dA = \iint_{R_{xy}} xy\sqrt{1+\frac{x^2+y^2}{x^2+y^2}}\,dA$$

$$= \sqrt{2}\iint_{R_{xy}} xy\,dA$$

convert to polar coordinates



$x = r\cos\theta \quad y = r\sin\theta$

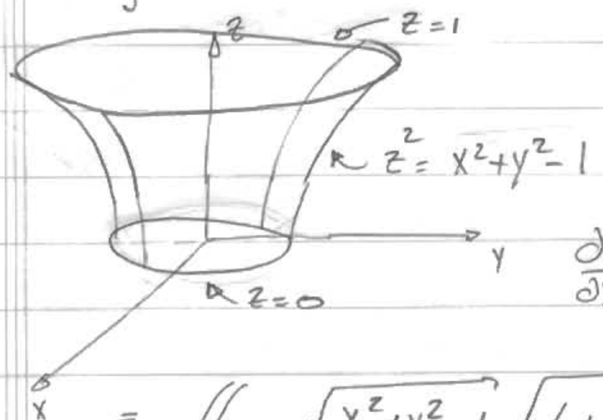
$$= \sqrt{2}\iint_R r^2\cos\theta\sin\theta\,dA$$

$$= \sqrt{2}\int_0^{\pi/2}\int_0^1 r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{\pi/2}\frac{r^4}{4}\Big|_0^1\cos\theta\sin\theta\,d\theta = \frac{\sqrt{2}}{4}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$

Schaums $$= \frac{\sqrt{2}}{4}\,\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = \frac{\sqrt{2}}{8}$$

c) $\iint_S z\,dS \qquad S:\ z^2 = x^2+y^2-1 \qquad z=0,\ z=1$



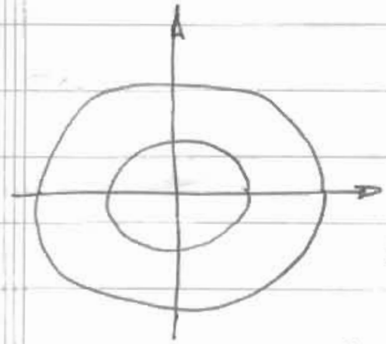
$\Rightarrow$ project onto xy plane

$z = \sqrt{x^2+y^2-1}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2-1}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2-1}}$$

$$= \iint_{R_{xy}} \sqrt{x^2+y^2-1}\sqrt{1+\frac{x^2+y^2}{x^2+y^2-1}}\,dA$$

$$= \iint_{R_{xy}} \cancel{\sqrt{x^2+y^2-1}}\sqrt{\frac{x^2+y^2-1+x^2+y^2}{\cancel{x^2+y^2-1}}} = \iint_{R_x}\sqrt{2x^2+2y^2-1}\,dA$$



evaluate in polar coordinates

$0 \le \theta \le 2\pi$

find intersection of $z = 1 = \sqrt{x^2+y^2-1}$

$x^2+y^2 = 2 \quad r = \sqrt{2}$

$\therefore \ 1 \le r \le \sqrt{2}$

$$\int_0^{2\pi}\int_1^{\sqrt{2}} \sqrt{2r^2\cos^2\theta + 2r^2\sin^2\theta - 1}\ r\,dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{2}} \sqrt{2r^2-1}\ r\,dr\,d\theta$$

substitution

$u = 2r^2-1$

$du = 4r\,dr$

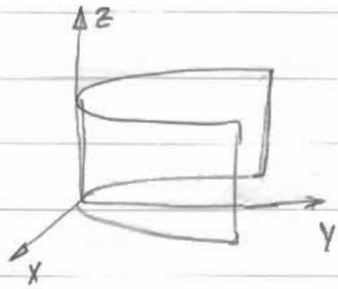
$r\,dr = \frac{du}{4}$

$\frac{1}{4}\int \sqrt{u}\,du = \frac{1}{4}\cdot\frac{2}{3}\cdot u^{3/2}$

$$= (2\pi)\left(\frac{1}{4}\cdot\frac{2}{3}(2r^2-1)^{3/2}\Big|_1^{\sqrt{2}}\right)$$

$$= \frac{\pi}{3}\left((2(2)-1)^{3/2} - (2-1)^{3/2}\right) = \frac{\pi}{3}\left(3\sqrt{3}-1\right)$$

2a) $\iint_S (yz^2, ye^x, x)\cdot\hat{n}\,dS \qquad S: \ y = x^2 \quad 0 \le y \le 4$
$\quad 0 \le z \le 1$



$\hat{n} \Rightarrow$ positive y component

$\vec{n} = \nabla(y-x^2) = (-2x, 1, 0)$

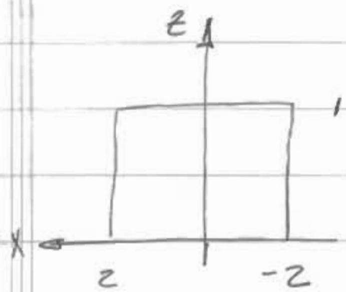
$$\hat{n} = \frac{(-2x, 1, 0)}{\sqrt{4x^2+1}}$$

project onto xz plane

$$= \iint_{R_{xz}} \frac{(yz^2)(-2x) + ye^x}{\sqrt{4x^2+1}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\ dA$$

substitute $y = x^2 \qquad \frac{\partial y}{\partial x} = 2x \quad \frac{\partial y}{\partial z} = 0$

$$= \iint_{R_{xz}} \frac{(x^2e^x - 2x^3z^2)}{\cancel{\sqrt{4x^2+1}}}\cancel{\sqrt{1+4x^2}}\, dA = \iint_{R_{xz}} x^2e^x - 2x^3z^2\, dA$$



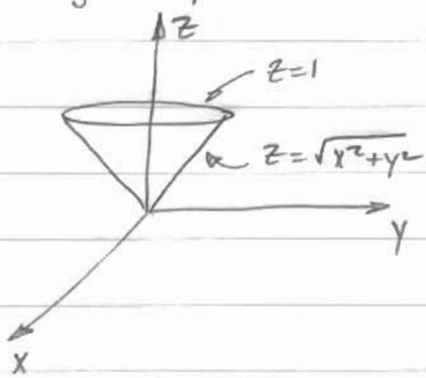
$$= \int_0^1 \int_{-2}^{2} x^2e^x - 2x^3z^2\, dx\, dz$$

Schaums $\int x^2e^x dx = e^x(x^2-2x+2)$

$$= \int_0^1 e^x(x^2-2x+2) - \frac{x^4}{2}z^2 \Big|_{-2}^{2} dz$$

$$= e^2(4-4+2) - e^{-2}(4+4+2) - 8z^2 + 8z^2 = 2e^2 - 10e^{-2}$$

b) $\iint_S (x,y,0)\cdot\hat{n}\, dS$ S: $z=\sqrt{x^2+y^2}$, $z=1$ (cone, plane)



$$\vec{n} = \nabla\left(\sqrt{x^2+y^2} - z\right)$$

$$= \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)$$

$$\hat{n} = \frac{\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)}{\left(\frac{x^2+y^2}{x^2+y^2}+1\right)^{1/2}}$$

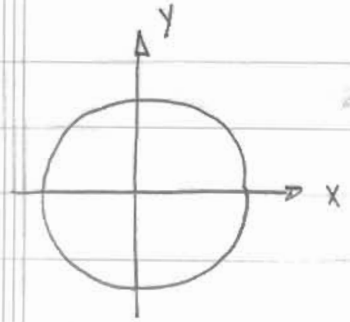
project on xy plane

$$\hat{n} = \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+y}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)$$

$$= \frac{1}{\sqrt{2}}\iint_{R_{xy}} \left(\frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}}\right)\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\; dA$$

$$= \iint_{R_{xy}} \sqrt{x^2+y^2}\, dA$$

⑤



projection on xy plane = circle

$\therefore$ polar coordinates

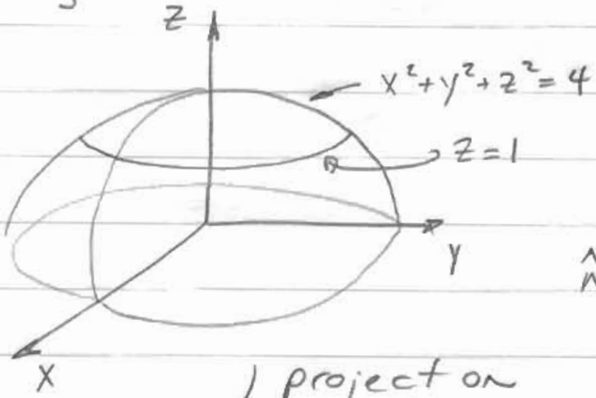
$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\sqrt{x^2+y^2} = r$$

$$= \int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta = \int_0^{2\pi} \frac{r^3}{3}\Big|_0^1 d\theta = \frac{2\pi}{3}$$

c) $\iint_S (x,y,0)\cdot\hat{n}\,dS \qquad S:\ x^2+y^2+z^2=4$ (sphere)

$z \geq 1$ (plane)



$$\vec{n} = \nabla(x^2+y^2+z^2-4)$$

$$= (2x, 2y, 2z)$$

$$\hat{n} = \frac{(2x,2y,2z)}{2\underbrace{\sqrt{x^2+y^2+z^2}}_{=4 \text{ for sphere}}} = \frac{(x,y,z)}{2}$$

project on xy plane

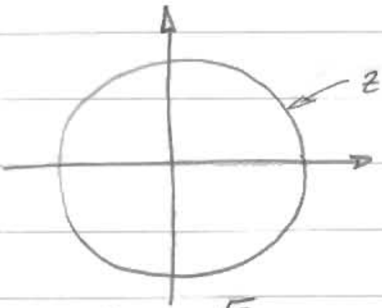
$$= \iint_{R_{xy}} \frac{(x^2+y^2)}{2}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \qquad z = \sqrt{4-x^2-y^2}$$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{4-x^2-y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{4-x^2-y^2}}$$

$$= \iint_{R_{xy}} \frac{(x^2+y^2)}{2}\sqrt{1+\frac{x^2+y^2}{4-x^2-y^2}}\,dA$$

$$= \iint_{R_{xy}} \frac{(x^2+y^2)}{2}\sqrt{\frac{4-x^2-y^2+x^2+y^2}{4-x^2-y^2}}\,dA = \iint_{R_{xy}} \frac{x^2+y^2}{\sqrt{4-x^2-y^2}}\,dA$$



$z = 1 = \sqrt{4-x^2-y^2}$

$x^2+y^2 = 3$

$r = \sqrt{3}$

polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{r^2}{\sqrt{4-r^2}}\, r\,dr\,d\theta$$

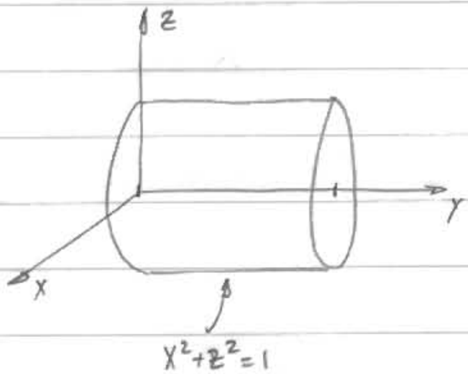
Schaums

$$\int \frac{x^3}{\sqrt{a^2-x^2}} = \frac{(a^2-x^2)^{3/2}}{3} - a^2\sqrt{a^2-x^2}$$

$$= (2\pi)\left[\frac{(4-r^2)^{3/2}}{3} - 4\sqrt{4-r^2}\,\Big|_0^{\sqrt{3}}\right]$$

$$= 2\pi\left(\frac{1}{3} - 4 - \frac{8}{3} + 8\right) = \frac{10\pi}{3}$$

3)



$$\vec{F} = e^{-y}\,\hat{\imath} + \frac{1}{y^2+1}\,\hat{\jmath}$$

$$\vec{n} = \nabla(x^2+z^2-1) = (2x, 0, 2z)$$

$$\hat{n} = \frac{(2x,0,2z)}{2\underbrace{\sqrt{x^2+z^2}}_{=1 \text{ on surface}}} = (x,0,z)$$

$$\vec{F} = \left(xe^{-y}, \frac{1}{y^2+1}, ze^{-y}\right)$$

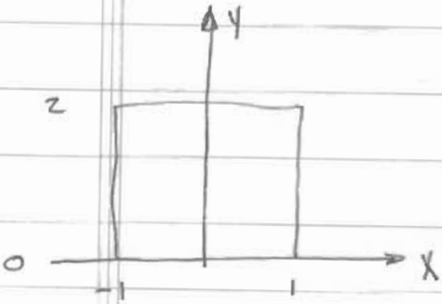
$$\vec{F}\cdot\hat{n} = \left(x^2e^{-y} + z^2e^{-y}\right) = (x^2+z^2)e^{-y} = e^{-y}$$

total outflow from surface $= \iint_{R_{xy}} e^{-y}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$

top half of cylinder $z = \sqrt{1-x^2}$ $\qquad \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1-x^2}} \qquad \frac{\partial z}{\partial y} = 0$

$$= \iint_{R_{xy}} e^{-y}\sqrt{1+\frac{x^2}{1-x^2}}\,dA = \iint_{R_{xy}} e^{-y}\sqrt{\frac{1}{1-x^2}}\,dA$$

$$= \iint_{R_{xy}} \frac{e^{-y}}{\sqrt{1-x^2}}\,dA$$



Schaums

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

$$= \int_0^2 \int_{-1}^1 \frac{e^{-y}}{\sqrt{1-x^2}}\,dx\,dy$$

$$= \int_0^2 e^{-y} \sin^{-1} x \Big|_{-1}^{1} dy = \left(\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right)\left(-e^{-y}\right)\Big|_0^2$$

$$= \pi\left(1 - e^{-2}\right)$$

$\therefore$ entire cylinder (top and bottom) $= 2\pi(1-e^{-2})$

4a) $\oiint_S (x^2, y^2, z^2)\cdot\hat{n}\,dS$ $\quad S:\ x^2+y^2+z^2 = a^2$

$\hat{n}$ = outward facing

divergence theorem

$$\oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V (\nabla\cdot\vec{F})\,dV \qquad \nabla\cdot\vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$$

$$= 2x + 2y + 2z$$

spherical coordinates

$x = r\sin\phi\cos\theta \qquad 2(x+y+z)$

$y = r\sin\phi\sin\theta \qquad = 2r(\sin\phi\cos\theta + \sin\phi\sin\theta + \cos\phi)$

$z = r\cos\phi$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 2r^2\left(\sin^2\phi\cos\theta + \sin^2\phi\sin\theta + \sin\phi\cos\phi\right) dr\,d\phi\,d\theta$$

$$= \frac{2a^3}{3}\int_0^{2\pi}\int_0^{\pi} \sin^2\phi(\cos\theta + \sin\theta) + \sin\phi\cos\phi\,d\phi\,d\theta$$

Schaums $\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4}$

$$\int \cos x \sin x\,dx = \frac{\sin^2 x}{2}$$

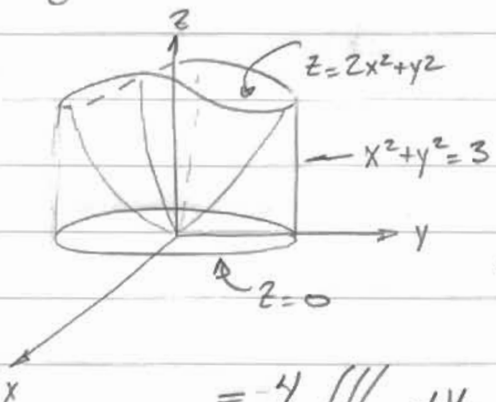
$$= \frac{2a^3}{3}\int_0^{2\pi}\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)(\cos\theta + \sin\theta) + \frac{\sin^2\theta}{2}\Big|_0^{\pi}\,d\theta$$

$$= \frac{2a^3}{3}\int_0^{2\pi}\frac{\pi}{2}(\cos\theta + \sin\theta)\,d\theta = \frac{2\pi a^3}{3}(\sin\theta - \cos\theta)\Big|_0^{2\pi}$$

$$= \frac{2\pi a^3}{3}(0 - 1 - 0 + 1) = 0$$

4b) $\oint\!\!\oint_S (x, y, 2z)\cdot\hat{n}\,dS$ S: $z = 2x^2 + y^2$ $x^2 + y^2 = 3$ $z = 0$

$\hat{n}$ = outward facing

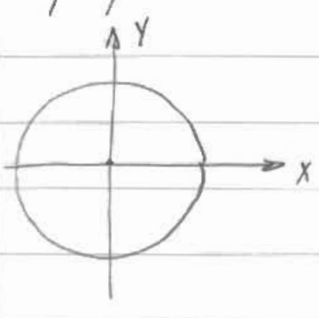


$= \iiint_V \nabla\cdot\vec{F}\,dV$

$\nabla\cdot\vec{F} = (1 + 1 + 2) = 4$

$= 4\iiint_V dV$ → integrate in z direction first $0 \le z \le 2x^2 + y^2$

project onto xy plane



→ easier to evaluate in polar coordinates

$x = r\cos\theta$ $y = r\sin\theta$

$0 \le z \le 2x^2 + y^2$

↳ $0 \le z \le 2r^2\cos^2\theta + r^2\sin^2\theta$

$\le r^2(\cos^2\theta + 1)$

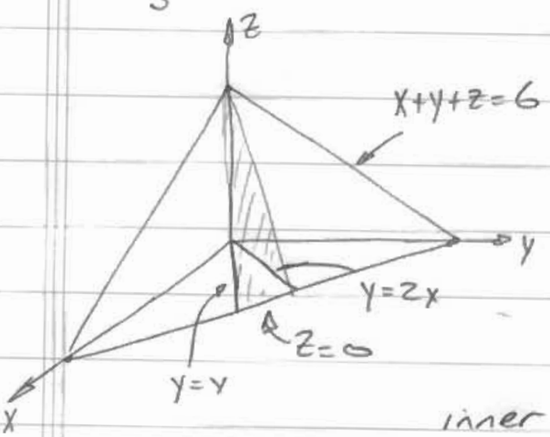
$$= 4\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{r^2(\cos^2\theta+1)} r\,dz\,dr\,d\theta = 4\int_0^{2\pi}\int_0^{\sqrt{3}} r^3(\cos^2\theta+1)\,dr\,d\theta$$

$$= 4\int_0^{2\pi}\frac{9}{4}(\cos^2\theta+1)\,d\theta = 9\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}+\theta\right)\Big|_0^{2\pi}$$

$$= 9\left(\frac{3}{2}\cdot 2\pi\right) = 27\pi$$

4c) $\oiint_S (xy, 0, z^2)\cdot\hat{n}\,dS$ S: $y=x$ $x+y+z=6$

$y=2x$ $z=0$



$\hat{n}$ = outward facing

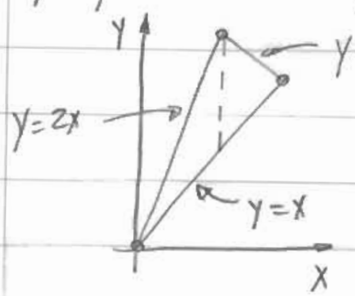
$= \iiint_V \nabla\cdot\vec{F}\,dV$

$\nabla\cdot\vec{F} = y + 0 + 2z = y + 2z$

inner integral in z direction

$0 \le z \le 6-x-y$

project onto xy plane



integrate in two parts

middle integrals

$x \le y \le 2x$, $x \le y \le 6-x$

find intersection points

$y = 2x = 6-x$ $y = x = 6-x$

$3x = 6$ $x = 2$ $2x = 6$ $x = 3$

Outer integrals $0 \le x \le 2$

$2 \le x \le 3$

$$= \int_0^2 \int_x^{2x} \int_0^{6-x-y} (y+2z)\,dz\,dy\,dx + \int_2^3 \int_x^{6-x} \int_0^{6-x-y} (y+2z)\,dz\,dy\,dx$$

$$= \int_0^2 \int_x^{2x} -6y + yx + 36 - 12x + x^2\,dy\,dx + \int_2^3 \int_x^{6-x} -6y + yx + 36 - 12x + x^2\,dy\,dx$$

$$= \int_0^2 \frac{5}{2}x^3 - 21x^2 + 36x\,dx + \int_2^3 -90x + 24x^2 - 2x^3 + 108\,dx$$

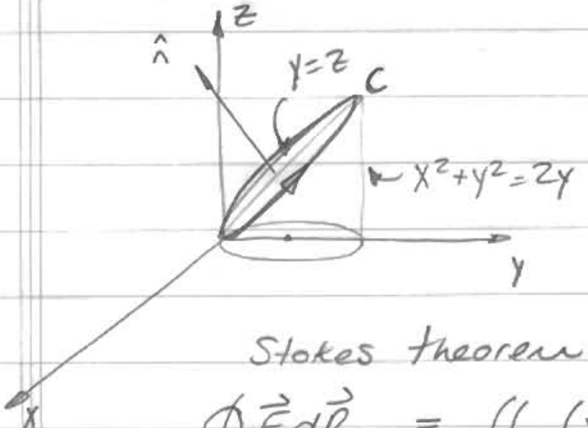
$$= \frac{5}{8}x^4 - 7x^3 + 18x^2 \Big|_0^2 + 45x^2 + 8x^3 - \frac{x^4}{4} + 108x \Big|_2^3 = \frac{57}{2}$$

5a) $\oint_C y^2dx + xy\,dy + xz\,dz$ C: $x^2+y^2=2y$

$$x^2 + y^2 - 2y + 1 = 1$$

$$\underbrace{x^2 + (y-1)^2 = 1}$$

circle, radius $=1$
center $(0,1)$



Stokes theorem

$$\oint_C \vec{F}\,d\vec{R} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & xy & xz \end{vmatrix} = (0-0)\hat{i} - (z-0)\hat{j} + (y-2y)\hat{k} = (0,-z,-y)$$

choose $S: y=z$ $\vec{n} = \nabla(z-y) = (0,-1,1)$

$$\hat{n} = \frac{(0,-1,1)}{\sqrt{2}}$$

$$(\nabla\times\vec{F})\cdot\hat{n} = \frac{1}{\sqrt{2}}(z-y)$$

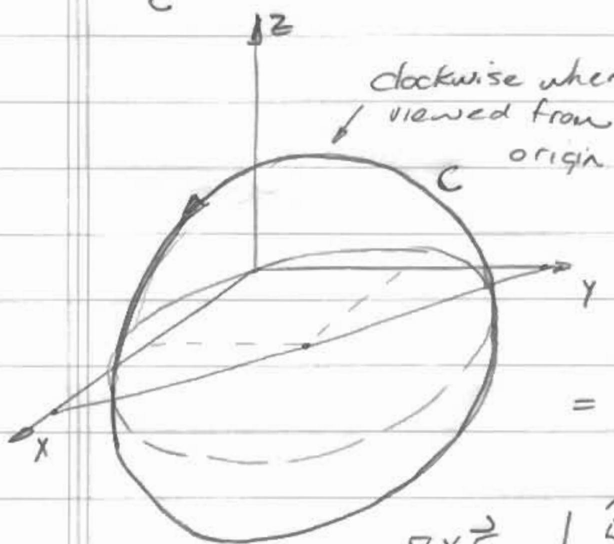
$$= \iint_S \frac{1}{\sqrt{2}}(z-y)\,dS = \frac{1}{\sqrt{2}}\iint_{R_{xy}} (z-y)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$z = y \quad \frac{\partial z}{\partial x} = 0 \quad \frac{\partial z}{\partial y} = 1$

$$= \frac{1}{\sqrt{2}}\iint_{R_{xy}} (y-y)\sqrt{1+1}\,dA = 0$$

5b) $\oint_C y\,dx + z\,dy + x\,dz$ $\qquad$ C: $x+y = 2b$ ← plane.

$$x^2+y^2+z^2 = 2bx + 2by$$
$$(x^2-2bx+b^2) + (y^2-2by+b^2) + z^2 = 2b^2$$
$$\underbrace{(x-b)^2 + (y-b)^2 + z^2 = 2b^2}_{\text{sphere, center } (b,b,0) \text{ radius } \sqrt{2}b}$$



$$= \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = (0-1)\hat{i} - (1-0)\hat{j} + (0-1)\hat{k} = (-1,-1,-1)$$

$$\vec{n} = \nabla(x+y-2b) = (1,1,0) \qquad \hat{n} = \frac{1}{\sqrt{2}}(1,1,0)$$

$$(\nabla\times\vec{F})\cdot\hat{n} = \frac{1}{\sqrt{2}}(-1-1) = -\sqrt{2}$$

$$\oint_C \vec{F}\,d\vec{R} = -\sqrt{2}\underbrace{\iint_S dS}$$

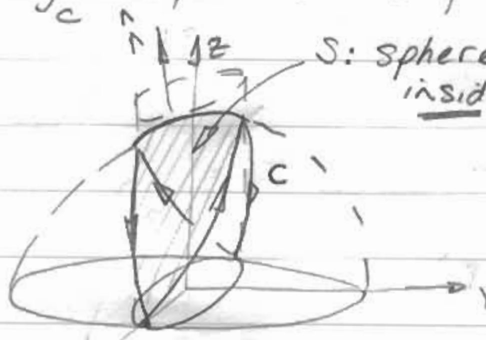
since S is a circle, radius $\sqrt{2}b$

$$\iint_S dS = \text{Area} = \pi(\sqrt{2}b)^2$$

$$\oint_C \vec{F}\cdot\vec{R} = -\sqrt{2}\pi(2b^2) = -2\sqrt{2}\pi b^2$$

5c) $\oint_c -2y^3x^2\,dx + x^3y^2\,dy + z\,dz$ $\quad$ C: $x^2+y^2+z^2=4$ (sphere)

$x^2+4y^2=4$ (elliptic cylinder)



S: sphere area <u>inside</u> C

C: Closed curve intersection of sphere and elliptic cylinder

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2y^3x^2 & x^3y^2 & z \end{vmatrix}$$

$$= (0-0)\hat{i} - (0-0)\hat{j} + (3x^2y^2 + 6x^2y^2)\hat{k}$$

$$\nabla\times\vec{F} = (0, 0, 9x^2y^2)$$

if S = any surface bounded by C

$\therefore$ Select $S = x^2+y^2+z^2=4$ $\leftarrow$ upper part of sphere bounded by C

$$\vec{n} = \nabla(x^2+y^2+z^2-4) = (2x, 2y, 2z)$$

$$\hat{n} = \frac{2(x,y,z)}{2\underbrace{\sqrt{x^2+y^2+z^2}}_{=4\text{ for sphere}}} = \frac{(x,y,z)}{2}$$

$$(\nabla\times\vec{F})\cdot\hat{n} = \frac{1}{2}(0+0+9x^2y^2z) = \frac{9}{2}x^2y^2z$$

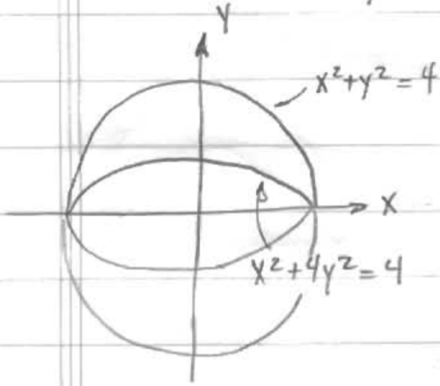
$$= \frac{9}{2}\iint_S x^2y^2z\,dS = \frac{9}{2}\iint_{R_{xy}} x^2y^2z\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$z = \sqrt{4-x^2-y^2}$$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{4-x^2-y^2}} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{4-x^2-y^2}}$$

$$= \frac{9}{2}\iint_{R_{xy}} x^2y^2\sqrt{4-x^2-y^2}\sqrt{1+\frac{x^2+y^2}{4-x^2-y^2}}\,dA$$

$$= \frac{9}{2}\iint_{R_{xy}} x^2y^2\sqrt{4-x^2-y^2}\sqrt{\frac{4}{4-x^2-y^2}}\,dA = 9\iint_{R_{xy}} x^2y^2\,dA$$



in first quadrant

$$= 36\int_0^1\int_0^{\sqrt{4-4y^2}} x^2y^2\,dx\,dy = \frac{36}{3}\int_0^1 x^3y^2\Big|_0^{\sqrt{4-4y^2}}\,dy$$

$$= 12\int_0^1 (4-4y^2)^{3/2}y^2\,dy = 96\int_0^1 y^2(1-y^2)^{3/2}\,dy$$

Schaums

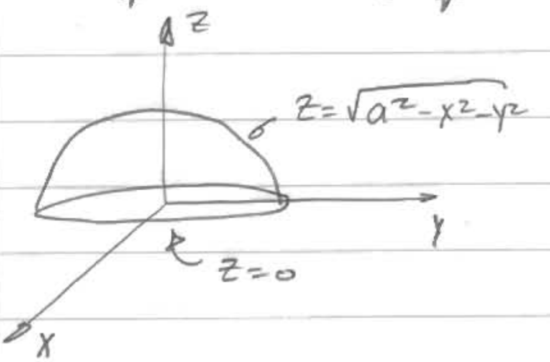
$$= 96\left[-\frac{y(1-y^2)^{5/2}}{6} + \frac{y(1-y^2)^{3/2}}{24} + \frac{y\sqrt{1-y^2}}{16} + \frac{\sin^{-1}y}{16}\right]\Bigg|_0^1$$

$$= 96\,\frac{\sin^{-1}(1)}{16} = 6\cdot\frac{\pi}{2} = 3\pi$$

6) $\oiint_S (xz^2,\ x^2y-z^3,\ 2xy+y^2z)\cdot\hat{n}\,dS$ S: $z=0$, $z=\sqrt{a^2-x^2-y^2}$

$\nabla\cdot\vec{F} = z^2+x^2+y^2$

$$= \iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V x^2+y^2+z^2\,dV$$



spherical coordinates

$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

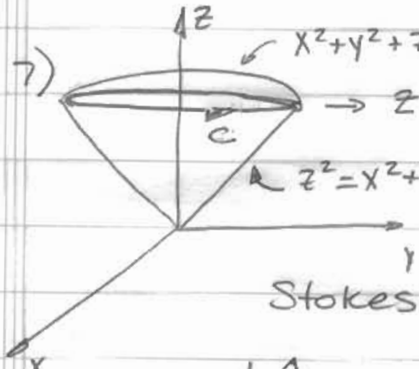
$z = r\cos\phi$

$$x^2+y^2+z^2 \Rightarrow r^2(\sin^2\phi(\cos^2\theta+\sin^2\theta)+\cos^2\phi) = r^2$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} r^4 \sin\phi \, dr\, d\phi\, d\theta = \frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \, d\phi\, d\theta$$

$$= \frac{a^5}{5}\int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi/2} d\theta = \left(\frac{a^5}{5}\right)(1)(2\pi) = \frac{2\pi a^5}{5}$$

7)



$\rightarrow z = \sqrt{1-x^2-y^2} = \sqrt{x^2+y^2}$

$x^2+y^2 = \left(\frac{1}{\sqrt{2}}\right)^2 \leftarrow$ radius of C

Stokes theorem $\oint_C \vec{F}\cdot d\vec{R} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+z & y^2+2x & z^2-y \end{vmatrix} = (-1-0)\hat{i} - (0-1)\hat{j} + (2-0)\hat{k} = (-1,1,2)$$

select plane surface $z=1$ for S, $\hat{n}=\hat{k}$

$$(\nabla\times\vec{F})\cdot\hat{n} = (-1)(0)+(1)(0)+(2)(1) = 2$$

$$= \iint_S 2\, dS = 2\iint_S dS$$

$\longrightarrow$ = area of circle

$$= 2\pi\left(\frac{1}{\sqrt{2}}\right)^2 = \pi$$