ASSIGNMENT #2 SOLUTIONS.

1a) $A(1,2,3)$ $\qquad \vec{AB} = (2,3,7)$

$B(3,5,10)$ $\qquad \vec{AC} = (-4,-6,-14)$

$C(-3,-4,-11)$ $\qquad \text{Area} = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 7 \\ -4 & -6 & -14 \end{vmatrix}$$

$$= \frac{1}{2}\left|(-42-(-42))\hat{i} - (-28-(-28))\hat{j} + (-12-(-12))\hat{k}\right|$$

$$= \frac{1}{2}\left|(0,0,0)\right| = 0$$

because $\vec{AB} = -\frac{1}{2}\vec{AC}$ $\therefore$ all three points lie on one line

1b) $A(1,-2,4)$ $\qquad \vec{AB} = (2,7,3)$

$B(3,5,7)$ $\qquad \vec{AC} = (3,8,4)$

$C(4,6,8)$

$\text{Area} = \left|\vec{AB} \times \vec{AC}\right|$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 7 & 3 \\ 3 & 8 & 4 \end{vmatrix} = \left|(28-24)\hat{i} - (8-9)\hat{j} + (16-21)\hat{k}\right|$$

$$= \sqrt{4^2+1^2+(-5)^2} = \sqrt{42}$$

2a) shortest distance - point to plane

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}$$

$P_1(-2,3,-5)$ $\qquad x_1 = -2$, $y_1 = 3$, $z_1 = -5$

plane $2x + y + 4z = 6$

$\therefore$ $A = 2$ $\quad C = 4$

$B = 1$ $\quad D = -6$

$$d = \frac{|(2)(-2) + (1)(3) + (4)(-5) - 6|}{\sqrt{2^2+1^2+4^2}} = \frac{27}{\sqrt{21}} \cong 5.892$$

2b) shortest distance - point to line

$$d = \frac{|\overrightarrow{P_1P_0} \times \overrightarrow{P_1P_2}|}{|\overrightarrow{P_1P_2}|}$$

$P_0\ (3,-2,0)$

line: $x = t$
$y = 3-2t$
$z = 4+t$

Select two points on line

$P_1 \Rightarrow x = 0 \quad t=0$
$y = 3-2(0) = 3$
$z = 4+0 = 4$

$P_2 \Rightarrow x = 5,\ t=5$
$y = 3-2(5) = -7$
$z = 4+5 = 9$

$P_1\ (0,3,4)$ $\quad$ $P_2\ (5,-7,9)$

$\overrightarrow{P_1P_0} = (3,-5,-4)$
$\overrightarrow{P_1P_2} = (5,-10,5)$

$$\overrightarrow{P_1P_0} \times \overrightarrow{P_1P_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -5 & -4 \\ 5 & -10 & 5 \end{vmatrix}$$

$$= (-25-(40))\hat{i} - (15-(-20))\hat{j} + (-30-(-25))\hat{k}$$

$$= (-65,-35,-5)$$

$$d = \frac{\sqrt{(65)^2+(-35)^2+(-5)^2}}{\sqrt{5^2+(-10)^2+5^2}} = \frac{\sqrt{146}}{2} \cong 6.042$$

2c) shortest distance - point $P_0\ (1,2,-3)$

line: $x = 2(y+1) = \frac{z-4}{2}$

select two points on line.

$P_1 \Rightarrow$ $x = 0$ $y = \frac{0-2}{2} = -1$ $z = 2(0)+4 = 4$ $P_2 \Rightarrow$ $x = 2$ $y = \frac{2-2}{2} = 0$ $z = 2(2)+4 = 8$ (3)

$P_1\ (0, -1, 4)$ $P_2\ (2, 0, 8)$

$\vec{P_1P_0} = (1, 3, -7)$ $\vec{P_1P_2} = (2, 1, 4)$

$$\vec{P_1P_0} \times \vec{P_1P_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -7 \\ 2 & 1 & 4 \end{vmatrix}$$

$$= (12-(-7))\hat{i} - (4-(-14))\hat{j} + (1-6)\hat{k} = (19, -18, -5)$$

$$d = \frac{\sqrt{19^2 + (-18)^2 + (-5)^2}}{\sqrt{2^2+1^2+4^2}} = \frac{\sqrt{14910}}{21} \cong 5.81$$

2d) shortest distance - lines $x = t$, $y = 3t-1$, $z = 1+2t$ $x = 2t+1$, $y = 1-t$, $z = 4+2t$

$$d = \frac{|\vec{RS} \cdot (\vec{PR} \times \vec{QS})|}{|\vec{PR} \times \vec{QS}|}$$

line 1 $\Rightarrow$ select two points

$t = 0$
$x = 0$
$y = 3(0) - 1 = -1$
$z = 1+2(0) = 1$
$P(0, -1, 1)$

$t = 1$
$x = 1$
$y = 3(1) - 1 = 2$
$z = 1+2(1) = 3$
$R(1, 2, 3)$

line 2 $\Rightarrow$ select two points

$t = 0$
$x = 2(0)+1 = 1$
$y = 1-0 = 1$
$z = 4+2(0) = 4$
$Q(1, 1, 4)$

$t = 1$
$x = 2(1)+1 = 3$
$y = 1-1 = 0$
$z = 4+2(1) = 6$
$S(3, 0, 6)$

$\vec{PR} = (1, 3, 2)$

$\vec{QS} = (2, -1, 2)$

$\vec{RS} = (2, -2, 3)$

$$\vec{PR} \times \vec{QS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ 2 & -1 & 2 \end{vmatrix} = (6-(-2))\hat{i} - (2-4)\hat{j} + (-1-6)\hat{k}$$ ④

$$= (8, 2, -7)$$

$$d = \frac{|(2,-2,3)\cdot(8,2,-7)|}{\sqrt{8^2+2^2+7^2}} = \frac{|16-4-21|}{\sqrt{117}} = \frac{9}{\sqrt{117}} = \frac{3\sqrt{13}}{13}$$

$$\cong 0.832$$

2e) shortest distance - lines

① $x+y-z=4$, $2x-z=4$

② $x = \frac{y+1}{2} = \frac{z-1}{3}$

convert line 1 to parametric form

assume $x = t$

$z = 2x-4 = 2t-4$

$y = 4-x+z$

$= 4-t+2t-4 = t$

$\therefore$ $x = t$, $y = t$, $z = 2t-4$

line 1 ⇒ select two points

$t=0$, $x=0$, $y=0$, $z=2(0)-4 = -4$ — $P(0,0,-4)$

$t=1$, $x=1$, $y=1$, $z=2(1)-4=-2$ — $R(1,1,-2)$

line 2 ⇒ select two points.

$Q(0,-1,1)$: $x=0$, $y=2(0)-1=-1$, $z=3(0)+1=1$

$S(1,1,4)$: $x=1$, $y=2(1)-1=1$, $z=3(1)+1=4$

$\vec{PR} = (1,1,2)$

$\vec{QS} = (1,2,3)$

$\vec{RS} = (0,0,6)$

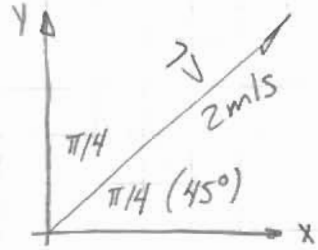
$$\vec{PR} \times \vec{QS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= (3-4)\hat{i} - (3-2)\hat{j} + (2-1)\hat{k}$$

$$= (-1,-1,1)$$

$$d = \frac{|(0,0,6)\cdot(-1,-1,1)|}{\sqrt{1^2+1^2+1^2}} = \frac{6}{\sqrt{3}} \cong 3.464$$

3a) parametric equations for balloon path



assume balloon speed = wind speed

$$V_x = |\vec{V}| \cdot \cos\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} \text{ m/s}$$

$$V_y = |\vec{V}| \cdot \sin\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} \text{ m/s}$$

V_z given $\quad V_z = 0.5$ m/s

parametric equations (position)

$$X = V_x \cdot t = \frac{2}{\sqrt{2}} t \quad [m]$$

$$y = V_y \cdot t = \frac{2}{\sqrt{2}} t \quad [m]$$

$$z = V_z \cdot t = 0.5t \quad [m]$$

3b) find shortest distance between two lines

① balloon flight path $\quad$ ② power line

② power line $\Rightarrow$ given coordinates $P(0, 75, 30)$

(assume no line sag) $\quad R(175, 175, 30)$

vector along power line $\quad \vec{PR} = (175, 100, 0)$

① balloon flight path $\Rightarrow$ select two points.

$t = 0$
$x = 0 \quad Q(0,0,0)$
$y = 0$
$z = 0$

$t = 10$
$x = 20/\sqrt{2} \quad S\left(\frac{20}{\sqrt{2}}, \frac{20}{\sqrt{2}}, 5\right)$
$y = 20\sqrt{2}$
$z = 5$

$$\vec{QS} = \left(\frac{20}{\sqrt{2}}, \frac{20}{\sqrt{2}}, 5\right) \qquad \vec{RS} = \left(\frac{20}{\sqrt{2}} - 175, \frac{20}{\sqrt{2}} - 175, -25\right)$$

$$\vec{PR} \times \vec{QS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 175 & 100 & 0 \\ \frac{20}{\sqrt{2}} & \frac{20}{\sqrt{2}} & 5 \end{vmatrix} = 500\hat{i} - (5)(175)\hat{j} + \frac{20}{\sqrt{2}}(75)\hat{k}$$

$$= \left(500, -875, \frac{1500}{\sqrt{2}}\right)$$

$$d = \frac{\left|\left(\frac{20}{\sqrt{2}}-175, \frac{20}{\sqrt{2}}-175, -25\right)\cdot\left(500, -875, \frac{1500}{\sqrt{2}}\right)\right|}{\sqrt{500^2+875^2+\frac{1500^2}{2}}}$$

$$d = \frac{65625 - 22500\sqrt{2}}{125\sqrt{137}} \cong 23.1 \text{ m.}$$

4a) moment of $\vec{F} = (3,-1,4)$ at $Q(1,1,0)$ about $P(2,1,-5)$

$$\vec{M} = \vec{PQ} \times \vec{F}$$

$$= (-1,0,5) \times (3,-1,4)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 5 \\ 3 & -1 & 4 \end{vmatrix} = (0-(-5))\hat{i} - (-4-15)\hat{j} + (1-0)\hat{k}$$

$$= (5, 19, 1)$$

$\therefore \quad \vec{M} = (5,19,1)$

b) component of moment vector of $\vec{F} = (6,-5,1)$ at $Q(-2,3,1)$ about line $\frac{x-3}{2} = y+1 = \frac{z}{4}$

express line as parametric equation

$\frac{x-3}{2} = t \qquad x = 2t+3 \qquad \frac{z}{4} = t \qquad z = 4t$

$y+1 = t \qquad y = t-1$

select two points along line

$t = 0$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $t = 1$

$x = 2(0) + 3 = 3$ $\quad\quad\quad$ $x = 2(1) + 3 = 5$

$y = 0 - 1 = -1$ $\quad\quad\quad\quad$ $y = 1 - 1 = 0$

$z = 4(0) = 0$ $\quad\quad\quad\quad\quad$ $z = 4(1) = 4$

$\hat{v}$ = unit vector along line

$$\vec{v} = (2, 1, 4) \qquad \hat{v} = \frac{(2,1,4)}{\sqrt{21}}$$

select one point on line $\quad P(5, 0, 4)$

$$\vec{PQ} = (-7, 3, -3)$$

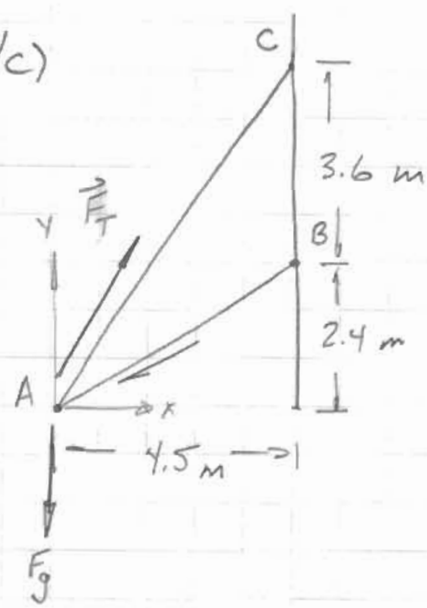
$$\vec{M} = \vec{PQ} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -7 & 3 & -3 \\ 6 & -5 & 1 \end{vmatrix} = (3 - 15)\hat{i} - (-7 - (-18))\hat{j} + (35 - 18)\hat{k}$$

$$= (-12, -11, 17)$$

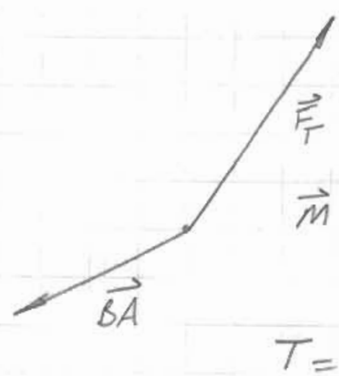
$$M_\ell = \vec{M} \cdot \hat{v} = (-12, -11, 17) \cdot \frac{(2,1,4)}{\sqrt{21}} = \frac{(-24 - 11 + 68)}{\sqrt{21}}$$

$$= 33/\sqrt{21} \cong 7.201$$

4c)



$\Rightarrow$



$$\vec{F}_T = T \cdot \hat{AC}$$

$$\vec{M} = \vec{BA} \times T \cdot \hat{AC}$$

$T = 1500\,N$

set origin at $A(0,0,0)$ $\quad$ [m]

$\therefore$ $B(4.5, 2.4, 0)$ $\quad$ [m]

$C(4.5, 6, 0)$ $\quad$ [m]

$\vec{BA} = (-4.5, -2.4, 0)$

$\vec{AC} = (4.5, 6, 0) \qquad \hat{AC} = \dfrac{(4.5, 6, 0)}{\sqrt{4.5^2 + 6^2}} = \dfrac{(4.5, 6, 0)}{(15/2)}$

$$\vec{m} = (-4.5, -2.4, 0) \times 1500\,(4.5, 6, 0) \cdot \frac{2}{15}$$

$$= (-4.5, -2.4, 0) \times (900, 1200, 0)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4.5 & -2.4 & 0 \\ 900 & 1200 & 0 \end{vmatrix}$$

$$= (0, 0, (-4.5)(1200) - (-2.4)(900)) = (0, 0, -3240)$$

$|\vec{m}| = 3240\ N \cdot m$

5a) $$\frac{d}{dt}\left(3(t, -t^2, 2t) + 4(1, -2t, 3t^2)\right)$$

$$= 3\frac{d}{dt}(t, -t^2, 2t) + 4\frac{d}{dt}(1, -2t, 3t^2)$$

$$= 3(1, -2t, 2) + 4(0, -2, 6t) = (3, -2(3t+4), 6(1+4t))$$

b) $$\int (t, -t^2, 2t)\,dt = \left(\frac{t^2}{2}, -\frac{t^3}{3}, t^2\right) + \vec{C}$$

where $\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$

and C_x, C_y, C_z = constants.

c) $$\frac{d}{dt}\left[t(\vec{u} \times \vec{v})\right] \qquad \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & -t^2 & 2t \\ 1 & -2t & 3t^2 \end{vmatrix}$$

$$= (-3t^4 - (-4t^2))\hat{i} - (3t^3 - 2t)\hat{j} + (-2t^2 - (-t^2))\hat{k}$$

$$= (4t^2 - 3t^4, 2t - 3t^3, -t^2)$$

$\frac{d}{dt}\left[t(4t^2-3t^4,\ 2t-3t^3,\ -t^2)\right] = \frac{d}{dt}(4t^3-3t^5,\ 2t^2-3t^4,\ -t^3)$

$$= (12t^2-15t^4,\ 4t-12t^3,\ -3t^2)$$

d) $\int [f(t)\ \vec{u}\cdot\vec{v}]\,dt$ $\quad \vec{u}\cdot\vec{v} = (t + (-t^2)(-2t) + (2t)(3t^2))$

$$= t + 2t^3 + 6t^3 = t + 8t^3$$

$$f(t)\cdot\vec{u}\cdot\vec{v} = (t^2+3)(t+8t^3)$$
$$= t^3 + 3t + 8t^5 + 24t^3$$
$$= 8t^5 + 25t^3 + 3t$$

$$\int f(t)\,\vec{u}\cdot\vec{v}\,dt = \int 8t^5 + 25t^3 + 3t\,dt = \frac{8}{6}t^6 + \frac{25}{4}t^4 + \frac{3}{2}t^2 + \vec{C}$$

6a) $\left.\begin{aligned} x &= t \\ y &= t^2 \\ z &= t^3 \\ t &\ge 0 \end{aligned}\right\}$ vector form

$$\vec{R} = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$
$$= t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$$

$$\vec{T} = \frac{d\vec{R}}{dt} = (1, 2t, 3t^2)$$

$$\hat{T} = \frac{(1, 2t, 3t^2)}{\sqrt{1+4t^2+9t^4}}$$

b) $x + y = 5$ $\quad (5, 0, 25)$ to $(0, 5, -5)$

$x^2 - y = z$ $\qquad -5 \le t \le 0$

select $x = -t$

← minus sign → x goes from 5 to 0 (decreases) as t increases from 0 to 5

solve $y = 5 - x$
$\quad = 5 + t$

$z = x^2 - y$
$\quad = t^2 - t - 5$

$x = -t$
$y = 5+t$
$z = t^2 - t - 5$

vector form

$\vec{R} = -t\,\hat{i} + (5+t)\,\hat{j} + (t^2-t-5)\,\hat{k}$

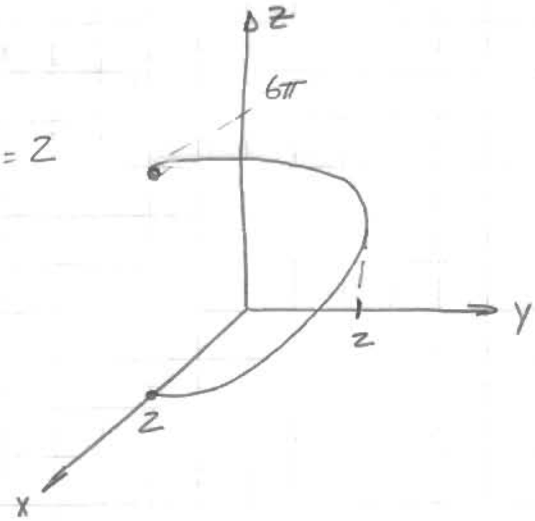
$-5 \le t \le 0$

$\vec{T} = \frac{dR}{dt} = (-1, 1, 2t-1)$

$$\hat{T} = \frac{(-1,1,2t-1)}{\sqrt{1+1+4t^2-4t+1}} = \frac{(-1,1,2t-1)}{\sqrt{4t^2-4t+3}}$$

7a) $x = 2\cos t$ $\quad 0 \le t \le 2\pi$
$y = 2\sin t$
$z = 3t$

helix - radius = 2
one rotation



$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

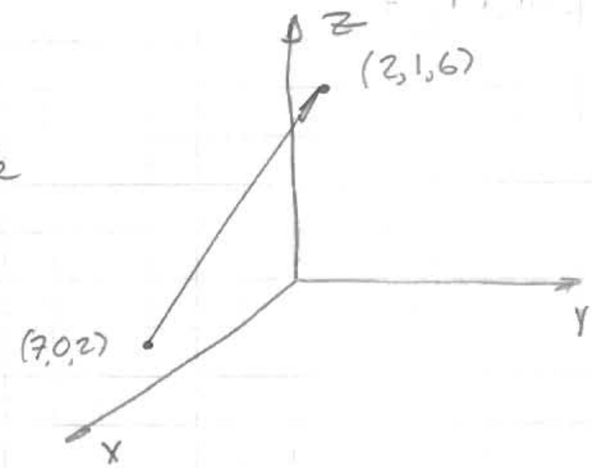
$$= \int_0^{2\pi} \sqrt{4(\underbrace{\sin^2 t + \cos^2 t}_{1}) + 9}\,dt$$

$$= \int_0^{2\pi} \sqrt{13}\,dt = 2\sqrt{13}\,\pi$$

t	x	y	z
-1	7	0	2
0	2	1	6

b) $x = 2 - 5t$ $\quad -1 \le t \le 0$
$y = 1 + t$
$z = 6 + 4t$

straight line



$\frac{dx}{dt} = -5$

$\frac{dy}{dt} = 1$

$\frac{dz}{dt} = 4$

$$L = \int_{-1}^{0} \sqrt{5^2 + 1^2 + 4^2}\,dt$$

$$= \sqrt{42}\,t \Big|_{-1}^{0}$$

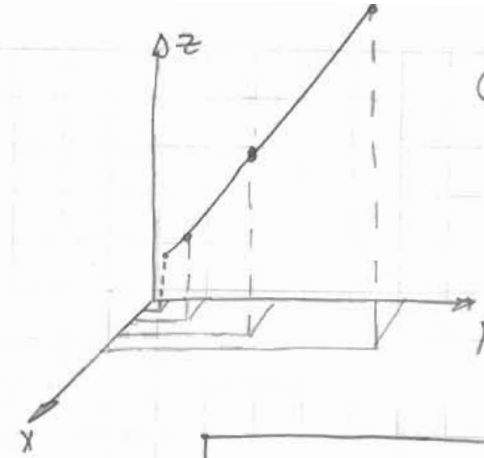
$$= \sqrt{42}\,(0-(-1)) = \sqrt{42} \cong 6.481$$

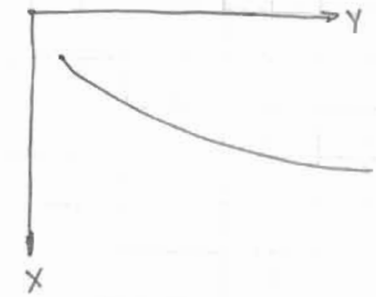
c) $x = t \qquad 1 \le t \le 4$

$y = t^{3/2}$

$z = 4t^{3/2}$

t	x	y	z
1	1	1	4
2	2	2.8	5.7
3	3	5.2	20.8
4	4	8	32





(11)

$dx/dt = 1$

$dy/dt = \frac{3}{2}\sqrt{t}$

$dz/dt = 6\sqrt{t}$ paraboloid

$$L = \int_1^4 \sqrt{1^2 + \frac{9}{4}t + 36t}\, dt$$

$$= \int_1^4 \sqrt{1 + \frac{153}{4}t}\, dt$$

let $w = 1 + \frac{153}{4}t$

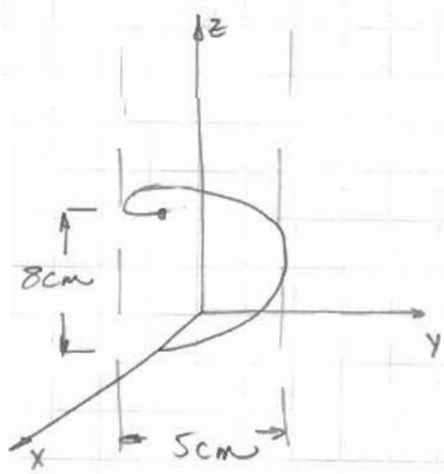
$dw = \frac{153}{4}dt \qquad dt = \frac{4}{153}dw$

$$\int \sqrt{w} \cdot \frac{4}{153}\, dw = \frac{4}{153} \cdot \frac{w^{3/2}}{(3/2)}$$

$$= \frac{8}{459}\left(1 + \frac{153}{4}t\right)^{3/2}$$

$$L = \frac{8}{459}\left(1 + \frac{153}{4}t\right)^{3/2}\Bigg|_1^4 \cong 29.02$$

8a)



wire forms a helix.

circular path in xy

$x = \frac{5}{2}\cos t \qquad 0 \le t \le 2\pi$

$y = \frac{5}{2}\sin t$

linear function in z

$$z = \frac{8t}{2\pi} = \frac{4t}{\pi}$$

b) 1 rotation $\Rightarrow$ $0 \le t \le 2\pi$, $0 \le z \le 8\text{cm}$

total # rotations (rad) for 2m length. $\dfrac{200\text{cm}}{8\text{cm}} \times \dfrac{2\pi}{\text{rot}} = 50\pi$

$\dfrac{dx}{dt} = -2.5\sin t$

$\dfrac{dy}{dt} = 2.5\cos t$

$\dfrac{dz}{dt} = \dfrac{4}{\pi}$

$$L = \int_0^{50\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \int_0^{50\pi} \sqrt{6.25(\underbrace{\sin^2 t + \cos^2 t}_{1}) + \frac{16}{\pi^2}}\, dt$$

$$= \int_0^{50\pi} \sqrt{6.25 + \frac{16}{\pi^2}}\, dt \cong (2.806)(50\pi)$$

$$= 440.7\text{ cm} = 4.41\text{ m}$$