ASSIGNMENT #4 SOLUTIONS.

1a) $$\frac{\partial^3}{\partial y^3}\left(\frac{2x}{y}+3x^3y^4\right) = \frac{\partial^2}{\partial y^2}\left(-\frac{2x}{y^2}+12x^3y^3\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{4x}{y^3}+36x^3y^2\right)$$

$$= \frac{-12x}{y^4}+72x^3y$$

b) $$\frac{\partial^2}{\partial y\partial z}\left(xyz\,e^{x+y+z}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}\left(xyz\,e^{x+y+z}\right)\right)$$

$$= \frac{\partial}{\partial y}\left(xy\left(e^{x+y+z}+z\,e^{x+y+z}\right)\right) = \frac{\partial}{\partial y}\left(xy\,e^{x+y+z}(1+z)\right)$$

$$= x\left(e^{x+y+z}+y\,e^{x+y+z}\right)(1+z) = x(1+y)(1+z)\,e^{x+y+z}$$

c) $$\frac{\partial^2}{\partial z^2}\ln\sqrt{x^2+y^2+z^2} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\ln\sqrt{x^2+y^2+z^2}\right] = \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\frac{1}{2}\,\frac{2z}{\sqrt{x^2+y^2+z^2}}\right]$$

$$= \frac{\partial}{\partial z}\left[\frac{z}{x^2+y^2+z^2}\right]$$

$$= \frac{1}{x^2+y^2+z^2} - \frac{2z^2}{(x^2+y^2+z^2)^2} = \frac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

d) $$\frac{\partial^6}{\partial x^2\partial y^2\partial z^2}\left[\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right] = \frac{\partial^5}{\partial x^2\partial y^2\partial z}\left[-\frac{2}{z^3}\right]$$

$$= \frac{\partial^4}{\partial x^2\partial y^2}\left[\frac{6}{z^4}\right] = 0$$

2) $z = x^2+xy+y^2\sin\left(\frac{x}{y}\right)$ $$\frac{\partial z}{\partial x} = 2x+y+y^2\cos\left(\frac{x}{y}\right)\cdot\frac{1}{y}$$

Check LHS expression → $$= 2x+y\left(1+\cos\left(\frac{x}{y}\right)\right)$$

$$\frac{\partial z}{\partial y} = x + 2y\sin\left(\frac{x}{y}\right) + y^2\cos\left(\frac{x}{y}\right)\cdot\left(\frac{-x}{y^2}\right)$$

$$= x + 2y\sin\left(\frac{x}{y}\right) - x\cos\left(\frac{x}{y}\right)$$

$$x\frac{\partial z}{\partial x} = 2x^2 + xy\left(1+\cos\left(\frac{x}{y}\right)\right)$$

$$y\frac{\partial z}{\partial y} = xy + 2y^2\sin\left(\frac{x}{y}\right) - xy\cos\left(\frac{x}{y}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2x^2 + xy + \cancel{xy\cos\left(\frac{x}{y}\right)} + xy + 2y^2\sin\left(\frac{x}{y}\right) - \cancel{xy\cos\left(\frac{x}{y}\right)}$$

$$= \underline{2\left(x^2 + xy + y^2\sin\left(\frac{x}{y}\right)\right) = 2z}$$

$\Rightarrow$ Check RHS expression.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x + y + y\cos\left(\frac{x}{y}\right)\right)$$

$$= 2 - y\cdot\sin\left(\frac{x}{y}\right)\cdot\frac{1}{y} = 2 - \sin\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x + 2y\sin\left(\frac{x}{y}\right) - x\cos\left(\frac{x}{y}\right)\right)$$

$$= 1 + 2\cancel{y}\cos\left(\frac{x}{y}\right)\cdot\frac{1}{\cancel{y}} - \cos\left(\frac{x}{y}\right) + x\sin\left(\frac{x}{y}\right)\cdot\frac{1}{y}$$

$$= 1 + \cos\left(\frac{x}{y}\right) + \frac{x}{y}\sin\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x + 2y\sin\left(\frac{x}{y}\right) - x\cos\left(\frac{x}{y}\right)\right)$$

$$= 2\sin\left(\frac{x}{y}\right) + 2y\cos\left(\frac{x}{y}\right)\cdot\left(\frac{-x}{y^2}\right) + x\sin\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)$$

$$= 2\sin\left(\frac{x}{y}\right) - \frac{2x}{y}\cos\left(\frac{x}{y}\right) - \frac{x^2}{y^2}\sin\left(\frac{x}{y}\right)$$

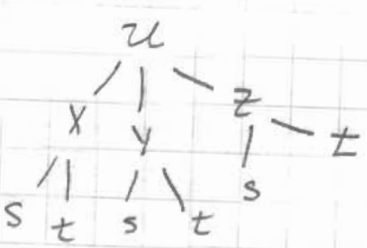
$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 2x^2 - \cancel{x^2\sin\left(\frac{x}{y}\right)} + 2xy$$

$$+ \cancel{2xy\cos\left(\frac{x}{y}\right)} + \cancel{2x^2\sin\left(\frac{x}{y}\right)} + 2y^2\sin\left(\frac{x}{y}\right) - \cancel{2xy\cos\left(\frac{x}{y}\right)} - \cancel{x^2\sin\left(\frac{x}{y}\right)}$$

$= 2x^2 + 2xy + 2y^2 \sin\left(\frac{x}{y}\right) = 2z \quad \therefore \quad LHS = RHS$ ③

3a) $u = \sqrt{x^2+y^2+z^2} \quad x = 2st \quad y = s^2+t^2 \quad z = st$

$\frac{\partial u}{\partial s} \rightarrow$



$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}$

$= \frac{x}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$

$\frac{\partial x}{\partial s} = 2t$

$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial y}{\partial s} = 2s$

$\frac{\partial u}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial z}{\partial s} = t$

$\frac{\partial u}{\partial s} = \frac{2tx}{\sqrt{x^2+y^2+z^2}} + \frac{2sy}{\sqrt{x^2+y^2+z^2}} + \frac{tz}{\sqrt{x^2+y^2+z^2}}$

$= \frac{2tx + 2sy + tz}{\sqrt{x^2+y^2+z^2}}$

$\sqrt{x^2+y^2+z^2} = \sqrt{4s^2t^2 + s^4 + 2s^2t^2 + t^4 + s^2t^2}$

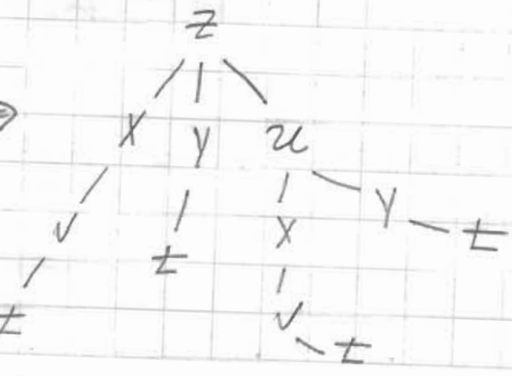
$= \sqrt{s^4 + 7s^2t^2 + t^4}$

$\therefore \frac{\partial u}{\partial s} = \frac{2st^2 + 2s^3 + 2st^2 + st^2}{\sqrt{s^4+7s^2t^2+t^4}} = \frac{2s^3 + 5st^2}{\sqrt{s^4+7s^2t^2+t^4}}$

3b) $z = x^2 + y^2 + u^2 \quad x = v^3 - 3v^2 \quad v = e^t$

$u = \frac{1}{x^2-y^2} \quad y = e^{4t}$

$\frac{dz}{dt} \Rightarrow$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y \qquad \frac{\partial z}{\partial u} = 2u$$

$$\frac{\partial x}{\partial v} = 3v^2 - 6v \qquad \frac{\partial v}{\partial t} = e^{t} \qquad \frac{\partial y}{\partial t} = 4e^{4t}$$

$$\frac{\partial u}{\partial x} = \frac{-1}{(x^2-y^2)^2}\cdot(2x) \qquad \frac{\partial u}{\partial y} = \frac{-1}{(x^2-y^2)^2}(-2y)$$

$$\frac{dz}{dt} = (2x)(3v^2-6v)(e^{t}) + (2y)(4e^{4t})$$

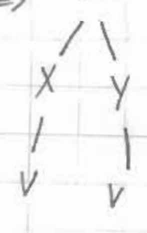
$$+ (2u)\left(\frac{-2x}{(x^2-y^2)^2}\right)(3v^2-6v)(e^{t}) + (2u)\left(\frac{2y}{(x^2-y^2)^2}\right)(4e^{4t})$$

$$= \left(2x + 2u\left(\frac{-2x}{(x^2-y^2)^2}\right)\right)(3v^2-6v)e^{t} + \left(2y + 2u\left(\frac{2y}{(x^2-y^2)^2}\right)\right)(4e^{4t})$$

$$\frac{dz}{dt} = 6xv(v-2)e^{t}\left(1 - \frac{2u}{(x^2-y^2)^2}\right) + 8ye^{4t}\left(1 + \frac{2u}{(x^2-y^2)^2}\right)$$

3c) $z = \sin(xy) \qquad x = 3\cos v \qquad y = 4\sin v$

$\frac{\partial z}{\partial v} \Rightarrow$



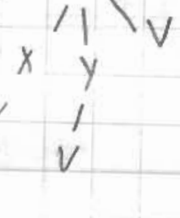
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dv} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dv}$$

$$= y\cos(xy)\cdot(-3\sin v) + x\cos(xy)\cdot(4\cos v)$$

$$= 4x\cos(xy)\cos(v) - 3y\cos(xy)\sin v$$

$$= \cos(xy)(4x\cos v - 3y\sin v)$$

$\frac{\partial^2 z}{\partial v^2} \Rightarrow$



$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\frac{dx}{dv} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\frac{dy}{dv} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)$$

5

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = -y\sin(xy)(4x\cos v - 3y\sin v) + \cos(xy)(4\cos v)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = -x\sin(xy)(4x\cos v - 3y\sin v) + \cos(xy)(-3\sin v)$$

$$\frac{\partial^2 z}{\partial v^2} = (4\cos v\cos(xy) - y\sin(xy)(4x\cos v - 3y\sin v))(-3\sin v)$$
$$-(3\sin v\cos(xy) + x\sin(xy)(4x\cos v - 3y\sin v))(4\cos v)$$
$$-4x\cos(xy)\sin v - 3y\cos(xy)\cos v$$
$$= -12\cos v\sin v\cos(xy) + 12xy\cos v\sin v\sin(xy) - 9y^2\sin^2 v\sin(xy)$$
$$-12\cos v\sin v\cos(xy) - 16x^2\cos^2 v\sin xy + 12xy\cos v\sin v\sin(xy)$$
$$-4x\sin v\cos(xy) - 3y\cos v\cos(xy)$$
$$= (-16x^2\cos^2 v + 24xy\cos v\sin v - 9y^2\sin^2 v)\sin(xy)$$
$$+ (-24\cos v\sin v - 4x\sin v - 3y\cos v)\cos(xy)$$

4 a) given curve $x = e^{-t}\cos t$
$y = e^{-t}\sin t$
$z = t$

$$\vec{R} = (e^{-t}\cos t,\ e^{-t}\sin t,\ t)$$

$$\vec{T} = \frac{d\vec{R}}{dt} = (-e^{-t}\cos t - e^{-t}\sin t,\ -e^{-t}\sin t + e^{-t}\cos t,\ 1)$$

at $(1,0,0)$ $t=0$

$$\vec{T}\Big|_{(1,0,0)} = (-1,1,1)$$

parametric equations $x = x_0 + t\,v_x$
$= 1 + t(-1) = 1-t$

$\therefore$ line $\Rightarrow$ $x = 1-t$
$y = t$
$z = t$

$y = y_0 + t\,v_y$
$= 0 + t(1) = t$

$z = z_0 + t\,v_z$
$= 0 + t(1) = t$

b) given two intersecting surfaces

$f = x^2+y^2+z^2-4$ $\quad \nabla f = (2x, 2y, 2z)$ $\quad \nabla f\Big|_{(1,1,-\sqrt{2})} = (2,2,-2\sqrt{2})$

$g = x^2+y^2-z^2$

$\nabla g = (2x, 2y, -2z)$ $\quad \nabla g\Big|_{(1,1,-\sqrt{2})} = (2,2,2\sqrt{2})$

tangent vector to curve at intersection from cross product of gradients

$\nabla f \times \nabla g$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & -2\sqrt{2} \\ 2 & 2 & 2\sqrt{2} \end{vmatrix} = (4\sqrt{2}+4\sqrt{2})\hat{i} - (4\sqrt{2}+4\sqrt{2})\hat{j} + (4-4)\hat{k}$$

$$= (8\sqrt{2}, -8\sqrt{2}, 0)$$

$$= (1,-1,0)$$ ← easier to work with.

$x = x_0 + t v_x \qquad x = 1 + t(1) = 1+t$

$y = y_0 + t v_y \qquad y = 1 + t(-1) = 1-t$

$z = z_0 + t v_z \qquad z = -\sqrt{2} + t(0) = -\sqrt{2}$

5 a) $f = x^2 - x - y^3 z$ at $(2,-1,-2)$

$\nabla f = (2x-1, -3y^2 z, -y^3)$

$\nabla f|_{(2,-1,-2)} = (3, 6, 1) = (A,B,C)$

plane $\Rightarrow$ $(A,B,C)\cdot(x-x_0, y-y_0, z-z_0) = 0$

$(3,6,1)\cdot(x-2, y+1, z+2) = 0$

$3x + 6y + z = 6 - 6 - 2 = -2$

$\boxed{\therefore\ 3x + 6y + z = -2}$

b) $f = x^2 + y^2 + 2y - 1$ at $(1,0,3)$

$\nabla f = (2x, 2y+2, 0)$

$\nabla f|_{(1,0,3)} = (2, 2, 0) = (A,B,C)$

$(2,2,0)\cdot(x-1, y-0, z-3) = 0$

$2x + 2y = 2 \qquad \boxed{x + y = 1}$

6 show $\nabla f_{surface} \perp$ to tangent vector of curve

$f = xyz - x^2 - 6y + 6$ ← NOTE TYPO IN QUESTION. change $+6y$ to $-6y$

$\nabla f = (yz - 2x, xz - 6, xy)$

$\nabla f|_{(1,1,1)} = (-1, -5, 1)$

tangent vector to intersection curve from cross product of gradients

$$g = x^2 - y^2 + z^2 - 1 \quad \nabla g = (2x, -2y, 2z) \quad \nabla g\Big|_{(1,1,1)} = (2,-2,2)$$

$$h = xy + xz - z \quad \nabla h = (y+z, x, z) \quad \nabla h\Big|_{(1,1,1)} = (2,1,1)$$

$$\vec{T} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = (-2-2)\hat{i} - (2-4)\hat{j} + (2+4)\hat{k}$$

$$= (-4, 2, 6)$$

$$\nabla f\Big|_{(1,1,1)} \cdot \vec{T}\Big|_{(1,1,1)} = (-1,-5,1)\cdot(-4,2,6) = 4 - 10 + 6 = 0$$

$\therefore$ vectors are $\perp$

7 a) for intersecting surfaces

$$f = x^2 + y^2 + z^2 - 2 \quad \nabla f = (2x, 2y, 2z) \quad \nabla f\Big|_{(0,1,1)} = (0,2,2)$$

$$g = y - z \quad \nabla g = (0, 1, -1)$$

$$\vec{T} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 2 \\ 0 & 1 & -1 \end{vmatrix} = (-2-2)\hat{i} - (0)\hat{j} + (0)\hat{k}$$

$$= (-4, 0, 0)$$

$$\hat{T} = \frac{(-4,0,0)}{\sqrt{16}} = (-1,0,0)$$

b) directional derivative in direction of increasing x

$\therefore$ change direction of $\hat{T} = (1,0,0)$

$$f = 2xyz - x^2 - z^2$$

$$\nabla f = (2yz - 2x, 2xz, 2xy - 2z)$$

$$\nabla f\Big|_{(0,1,1)} = (2, 0, -2)$$

$D_{\hat{v}} f = (2, 0, -2) \cdot (1, 0, 0) = 2$

c) find angle θ

$$D_{\hat{v}} f = |\nabla f||\hat{v}| \cos\theta \quad \text{where } |\hat{v}| = 1$$

$$|\nabla f| \cos\theta = 0 \qquad \therefore \ \theta = \pi/2$$

$$|\nabla f| \cos\theta = 1 \qquad \theta = \cos^{-1}\left(\frac{1}{|\nabla f|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{4+4}}\right)$$

$$= \cos^{-1}\left(\frac{1}{2\sqrt{2}}\right) \cong 1.21 \text{ rad}$$

$|\nabla f| \cos\theta =$ maximum when $\theta = 0$