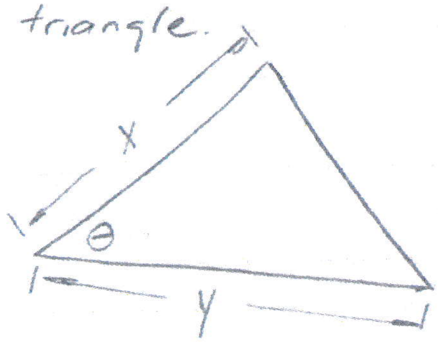
1 triangle.




$A = \frac{1}{2} xy \sin\theta$ gvn $\frac{dx}{dt} = 0.5$ m/s

$\frac{dy}{dt} = 0.5$ m/s

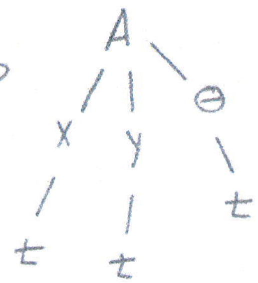
$\frac{d\theta}{dt} = -0.1$ rad/s

decreases

find rate of change of $A$

$$\left.\frac{\partial A}{\partial t}\right|_{x=1,\ y=2,\ \theta=\frac{1}{3}}$$

$\frac{\partial A}{\partial t} \Rightarrow$




$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\cdot\frac{d\theta}{dt}$$

$$= \left(\frac{y}{2}\sin\theta\right)\frac{dx}{dt} + \left(\frac{x}{2}\sin\theta\right)\frac{dy}{dt} + \left(\frac{xy}{2}\cos\theta\right)\frac{d\theta}{dt}$$

substitute values for $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$

$$\frac{\partial A}{\partial t} = \frac{y}{4}\sin\theta + \frac{x}{4}\sin\theta - \frac{xy}{20}\cos\theta$$

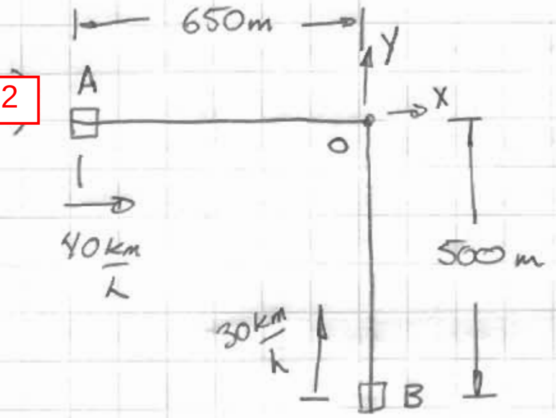
$$= \frac{\sin\theta}{4}(x+y) - \frac{\cos\theta}{20}(xy)$$

evaluate for $x = 1$ m, $y = 2$ m, $\theta = 1/3$ rad.

$$\frac{\partial A}{\partial t} = \frac{\sin(1/3)}{4}(3) - \frac{\cos(1/3)}{20}(2)$$

$$\cong 0.151 \ \frac{m^2}{s}$$

2




a) cartesian coordinates, origin at intersection

b) find rate of change of distance between CARS A and B

$D = \sqrt{x^2 + y^2}$ where $X$ = distance from A to origin

$Y$ = distance from B to origin

find $\frac{\partial D}{\partial t} = \frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt}$

$$\begin{array}{ccc} & D & \\ / & & \backslash \\ X & & Y \\ | & & | \\ t & & t \end{array}$$

$$\frac{\partial D}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} \qquad \frac{\partial D}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{dx}{dt} = 40 \qquad \frac{dy}{dt} = 30$$

$$\frac{\partial D}{\partial t} = \frac{40x + 30y}{\sqrt{x^2+y^2}}$$

$$\left.\frac{\partial D}{\partial t}\right|_{x=-0.65,\ y=-0.5} = \frac{40(-0.65)+30(-0.5)}{\sqrt{(0.65)^2+(0.5)^2}} = -49.996$$

$$\cong -50 \text{ km/h}$$

$\therefore$ distance decreases at 50 km/h.

**3 a)** $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ $\quad f = x^2y + xy^2$

$$\frac{\partial f}{\partial x} = 2xy + y^2$$

$$\frac{\partial f}{\partial y} = x^2 + 2xy$$

$$\nabla f = (2xy + y^2)\hat{i} + (x^2 + 2xy)\hat{j}$$

b) $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ $\quad f = e^{x+y+z}$

$$\frac{\partial f}{\partial x} = e^{x+y+z}$$

$$\frac{\partial f}{\partial y} = e^{x+y+z} = \frac{\partial f}{\partial z}$$

$$\therefore \nabla f = e^{x+y+z}(\hat{i} + \hat{j} + \hat{k})$$

c) $f = xy \ln(x+y)$ $\quad \frac{\partial f}{\partial x} = y\ln(x+y) + \frac{xy}{x+y}$

$$\frac{\partial f}{\partial y} = x\ln(x+y) + \frac{xy}{x+y}$$

$$\nabla f = \left(y\ln(x+y) + \frac{xy}{x+y}\right)\hat{i} + \left(x\ln(x+y) + \frac{xy}{x+y}\right)\hat{j}$$

evaluate at $(4,-2)$

$$\nabla f = \left(-2\ln 2 - \frac{8}{2}\right)\hat{i} + \left(4\ln 2 - \frac{8}{2}\right)\hat{j}$$

$$= -2(\ln 2 + 2)\hat{i} + 4(\ln 2 - 1)\hat{j} \cong (-5.386, -1.227)$$

**4 a)** find $D_{\vec{v}}f$ for $f = \ln(xy + yz + xz)$

$\vec{v}$ = vector from $(1,1,1)$ to $(-1,-2,3)$

$$\vec{v} = (-1-1, -2-1, 3-1) = (-2,-3,2)$$

$D_{\vec{v}}f = \nabla f \cdot \hat{v}$ $\rightarrow$ find $\nabla f\big|_{(1,1,1)}$ and $\hat{v}$

$$\nabla f = \left(\frac{y+z}{xy+yz+xz}\right)\hat{i} + \left(\frac{x+z}{xy+yz+xz}\right)\hat{j} + \left(\frac{y+z}{xy+yz+xz}\right)\hat{k}$$

$$\nabla f\Big|_{(1,1,1)} = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{(-2,-3,2)}{\sqrt{4+9+4}} = \frac{1}{\sqrt{17}}(-2,-3,2)$$

$$\therefore D_{\hat{v}} f = \frac{1}{\sqrt{17}}\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)\cdot(-2,-3,2) = \frac{1}{\sqrt{17}}\left(-\frac{4}{3} - 2 + \frac{4}{3}\right)$$

$$= -\frac{2}{\sqrt{17}} \cong -0.485$$

b) $f = x^2y + y^2z + z^2x$

$$\nabla f = (2xy + z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + 2zx)\hat{k}$$

$$\nabla f\Big|_{(1,-1,0)} = (-2, 1, 1)$$

$\vec{v}$ from intersection of surfaces $x + 2y + 1 = 0$, $x - y + 2z = 2$

let $x = t$

solve eq #1 $\quad t + 2y + 1 = 0 \quad y = -\frac{(t+1)}{2} = -\frac{1}{2}(t+1)$

solve eq #2 $\quad t + \frac{t+1}{2} + 2z = 2 \quad z = \frac{2}{2} - \frac{t}{2} - \frac{t}{4} - \frac{1}{4}$

$\therefore$ line $\quad x = t \qquad z = \frac{3}{4} - \frac{3}{4}t = \frac{3}{4}(1-t)$

$y = -\frac{1}{2}(t+1)$

$z = \frac{3}{4}(1-t) \leftarrow$ as $t$ increases, $z$ decreases

$\therefore$ "in direction of decreasing $z$"

since $t = 1$ for $(1,-1,0)$

evaluate $t = 2 \rightarrow$ gives point $\left(2, -\frac{3}{2}, -\frac{3}{4}\right)$

$$\vec{v} = \left(2-1, -\frac{3}{2} - (-1), -\frac{3}{4} - 0\right) = \left(1, -\frac{1}{2}, -\frac{3}{4}\right)$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{(1,-\frac{1}{2},-\frac{3}{4})}{\sqrt{1+\frac{1}{4}+\frac{9}{16}}} = \frac{(1,-\frac{1}{2},-\frac{3}{4})}{\frac{\sqrt{29}}{\sqrt{16}}} = \frac{(4,-2,-3)}{\sqrt{29}}$$

$$\therefore D_{\vec{v}} f = \frac{1}{\sqrt{29}}(-2,1,1)\cdot(4,-2,-3) = \frac{(-8-2-3)}{\sqrt{29}} = \frac{-13}{\sqrt{29}} \cong -2.414$$

c) $f = x^2 + y$

$$\nabla f = 2x\,\hat{i} + \hat{j} \qquad \nabla f\big|_{(-1,3)} = (-2,1)$$

$\vec{v}$ from tangent vector to curve $y = -3x^3$ at $(-1,3)$

$-3x^3 = y = t$ ← symmetric form.

$$\therefore \quad x = \left(-\frac{t}{3}\right)^{1/3}$$

$$y = t$$

$$\vec{R} = \left(\left(-\frac{t}{3}\right)^{1/3}, t\right)$$

$$\boxed{\begin{array}{l}\underline{\text{assume}} \\ (-1)^{2/3} = \left((-1)^2\right)^{1/3} \\ \qquad = 1^{1/3} = 1\end{array}}$$

$$\vec{T} = \frac{d\vec{R}}{dt} = \left(\frac{1}{3}\cdot\left(-\frac{t}{3}\right)^{-\frac{2}{3}}\left(-\frac{1}{3}\right), 1\right) = \left(-\frac{1}{9}\left(\frac{3}{-t}\right)^{\frac{2}{3}}, 1\right)$$

at $(-1,3)$ $\quad t = 3$

$$\therefore \vec{T} = \left(\frac{-1}{9}(-1)^{\frac{2}{3}}, 1\right) = \left(\frac{-1}{9}, 1\right)$$

direction of tangent vector is
in negative $x$ - direction is correct.

$$\hat{T} = \frac{(-\frac{1}{9}, 1)}{\sqrt{\frac{1}{81}+1}} = \frac{(-\frac{1}{9}, 1)}{\frac{\sqrt{82}}{\sqrt{81}}} = \frac{(-1,9)}{\sqrt{82}}$$

$$D_{\hat{T}} f = \frac{1}{\sqrt{82}}(-2,1)\cdot(-1,9) = \frac{11}{\sqrt{82}} \cong 1.215$$

5 by definition $D_{\vec{v}} f$ increases most rapidly when $\vec{v}$ is in the $\nabla f$ direction (10)

$\therefore$ direction $= \nabla f = \nabla(\tan^{-1}(xyz))$

from Schaum's (pg. 60) $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$

$\therefore \frac{\partial f}{\partial x} = \frac{1}{1+(xyz)^2} \cdot yz \qquad \frac{\partial f}{\partial y} = \frac{xz}{1+(xyz)^2}$

$\frac{\partial f}{\partial z} = \frac{xy}{1+(xyz)^2}$

$\therefore \nabla f = \left(\frac{yz}{1+(xyz)^2}\right)\hat{i} + \left(\frac{xz}{1+(xyz)^2}\right)\hat{j} + \left(\frac{xy}{1+(xyz)^2}\right)\hat{k}$

$\nabla f\big|_{(3,2,-4)} = \frac{1}{1+24^2}\left(-8\hat{i} - 12\hat{j} + 6\hat{k}\right) = \frac{(-8,-12,6)}{577}$

by definition – in $\nabla f$ direction

$D_{\vec{v}} f = |\nabla f| \cdot |\hat{v}| \cdot \cos\theta$ where $\theta = 0$

$|\hat{v}| = 1$

$\therefore$ rate of change $= |\nabla f| = \frac{\sqrt{64+144+36}}{577} = \frac{2\sqrt{61}}{577}$

$\cong 0.0271$

6 from definition of directional derivative

$D_{\vec{v}} f = |\nabla f|\,|\hat{v}| \cos\theta$ = smallest when $\theta = \pi/2$

$= \perp$ to $\nabla f$

$f = xyz \qquad \nabla f = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k} \qquad \nabla f\big|_{(2,-1,3)} = (-3,6,-2)$

find equation for plane tangent to surface $f = xyz$ at $(2,-1,3)$

$(A,B,C)\cdot(x-x_0,\ y-y_0,\ z-z_0)=0$

$-3(x-2)+6(y+1)-2(z-3)=0$

$-3x+6y-2z=-18$

$3x-6y+2z=18$ $\therefore$ $\vec{v}$ from two points of surface

$(2,-1,3)$ to $(0,0,9)$

$\vec{v}=(-2,1,6)$

check if $\nabla f$ and $\vec{v}\perp$ $\nabla f\big|_{(2,-1,3)}\cdot\vec{v}=(-3,6,-2)\cdot(-2,1,6)$

$=6+6-12=0$

$D_{\vec{v}}f=0$

<u>alternate answer</u> $D_{\vec{v}}f$ smallest (most negative) in $-\nabla f$ direction

if $\nabla f\big|_{(2,-1,3)}=(-3,6,-2)$ then $-\nabla f\big|_{(2,-1,3)}=(3,-6,2)$

7 a) if $D_{\hat{v}}f=0$ given $f(x,y)=x^2y+y^3$ at $(1,-1)$

find $\hat{v}$

if $D_{\hat{v}}f=0$ then $\nabla f\big|_{(1,-1)}$ is $\perp$ to $\hat{v}$

$f=x^2y+y^3$ $\nabla f=2xy\,\hat{i}+(x^2+3y^2)\hat{j}$

$\nabla f\big|_{(1,-1)}=(-2,4)$

$\nabla f\big|_{(1,-1)}\cdot\hat{v}=(-2,4)\cdot(v_x,v_y)=0$

$-2v_x+4v_y=0$

select $v_x=1$ solve $v_y=\dfrac{2(1)}{4}=\dfrac{1}{2}$

$\therefore$ $\vec{v}=(1,\tfrac{1}{2})$ or $(-1,-\tfrac{1}{2})$ $\hat{v}=\dfrac{(2,1)}{\sqrt{5}}$ or $\dfrac{(-2,-1)}{\sqrt{5}}$

b) if $D_{\hat{v}} f = 1$ and $\nabla f \big|_{(1,-1)} = (-2, 4)$

find $\hat{v}$

$$D_{\hat{v}} f = \nabla f \cdot \hat{v} = (-2, 4) \cdot (v_x, v_y) = 1$$

$$-2v_x + 4v_y = 1$$

since $\hat{v}$ has unit length $\quad |\hat{v}| = \sqrt{v_x^2 + v_y^2} = 1$

$$v_x^2 + v_y^2 = 1$$

$\therefore$ two equations, two unknowns

$$-2v_x + 4v_y = 1 \qquad v_y = \frac{1 + 2v_x}{4}$$

$$v_x^2 + v_y^2 = 1 \qquad v_y^2 = 1 - v_x^2 = \frac{1 + 4v_x + 4v_x^2}{16}$$

$$16 - 16v_x^2 = 1 + 4v_x + 4v_x^2$$

$$20v_x^2 + 4v_x - 15 = 0$$

$$v_x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4(20)(-15)}}{40} = \frac{-4 \pm 8\sqrt{19}}{40} = \frac{-1 \pm 2\sqrt{19}}{10}$$

$$v_y = \frac{1}{4} + \frac{2}{4}\left(\frac{-1 \pm 2\sqrt{19}}{10}\right) = \frac{10}{40} - \frac{2}{40} \pm \frac{4\sqrt{19}}{40} = \frac{8 \pm 4\sqrt{19}}{40} = \frac{2 \pm \sqrt{19}}{10}$$

$$\therefore \hat{v} = \left(\frac{-1 \pm 2\sqrt{19}}{10}, \frac{2 \pm \sqrt{19}}{10}\right)$$

c) if $D_{\hat{v}} f = 20$ and $\nabla f \big|_{(1,-1)} = (-2, 4)$

two equations

$$-2v_x + 4v_y = 20 \qquad v_y = \frac{20 + 2v_x}{4} = 5 + \frac{1}{2}v_x$$

$$v_x^2 + v_y^2 = 1$$

$$v_y^2 = 1 - v_x^2 = 25 + 5v_x + \frac{v_x^2}{4} \qquad \frac{5}{4}v_x^2 + 5v_x + 24 = 0$$

$$5v_x^2 + 20v_x + 96 = 0$$

$$V_x = \frac{-20 \pm \sqrt{20^2 - 4(5)(96)}}{10} = \frac{-20 \pm \sqrt{400 - 1920}}{10}$$ (13)

negative value ∴ complex roots

since $V_x$, $V_y$, $\hat{v}$ undefined then

$D_{\hat{v}} f$ cannot equal 20

8 distance to origin $D = \sqrt{x^2+y^2+z^2}$

find rate of change of $D$ along $x = 2\cos t$, $y = 2\sin t$, $z = 3t$

$= D_{\hat{T}} f$

$$f = \sqrt{x^2+y^2+z^2}$$

$$\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

along curve

$\vec{R} = (2\cos t, 2\sin t, 3t)$

$\vec{T} = (-2\sin t, 2\cos t, 3)$ $\quad \hat{T} = \dfrac{(-2\sin t, 2\cos t, 3)}{\sqrt{4(\sin^2 t + \cos^2 t) + 9}}$

$$= \frac{1}{\sqrt{13}}(-2\sin t, 2\cos t, 3)$$

$$D_{\hat{T}} f = \frac{1}{\sqrt{x^2+y^2+z^2}}(x, y, z) \cdot (-2\sin t, 2\cos t, 3) \cdot \frac{1}{\sqrt{13}}$$

$$= \frac{1}{\sqrt{4(\sin^2 t + \cos^2 t) + 9t^2}} \cdot \frac{(-4\sin t\cos t + 4\cos t\sin t + 9t)}{\sqrt{13}}$$

$$= \frac{9t}{\sqrt{13}\sqrt{4+9t^2}}$$

when $t = 0$ $\quad D_{\hat{T}} f = 0$

minimum distance to origin when $t=0$, ∴ derivative $=0$