Assignment #7

1 a) $$\int_{-3}^{3}\int_{-\sqrt{18-2y^2}}^{\sqrt{18-2y^2}} x\,dx\,dy = \int_{-3}^{3} \frac{x^2}{2}\Big|_{-\sqrt{18-2y^2}}^{\sqrt{18-2y^2}} dy$$

$$= \frac{1}{2}\int_{-3}^{3} 18-2y^2-(18-2y^2)\,dy = \frac{1}{2}\int_{-3}^{3} 0\,dy = 0$$

b) $$\int_{-1}^{0}\int_{y}^{2}(1+y)^2\,dx\,dy = \int_{-1}^{0} x(1+y)^2\Big|_{y}^{2} dy$$

$$= \int_{-1}^{0} 2(1+y)^2 - y(1+y)^2\,dy$$

$$= \int_{-1}^{0} 2+4y+2y^2-y-2y^2-y^3\,dy$$

$$= \int_{-1}^{0} 2+3y-y^3\,dy = 2y+\frac{3}{2}y^2-\frac{y^4}{4}\Big|_{-1}^{0}$$

$$= -\left(-2+\frac{3}{2}-\frac{1}{4}\right) = \frac{3}{4}$$

c) $$\int_{1}^{2}\int_{1}^{y} e^{x+y}\,dx\,dy \Rightarrow \text{since } y = \text{constant} \therefore \text{inner integral}$$

$$= \int_{1}^{2} e^{x+y}\Big|_{1}^{y} dy = \int_{1}^{2} e^{2y}-e^{1+y}\,dy$$

$$= \frac{1}{2}e^{2y}-e^{1+y}\Big|_{1}^{2} = \frac{1}{2}e^4-e^3-\left(\frac{1}{2}e^2-e^2\right)$$

$$= \frac{1}{2}e^4-e^3+\frac{1}{2}e^2 = \frac{e^2}{2}\left(e^2-2e+1\right)$$

d) $$\int_{-1}^{1}\int_{x}^{2x}(xy+x^3y^3)\,dy\,dx = \int_{-1}^{1}\frac{xy^2}{2}+\frac{x^3y^4}{4}\Big|_{x}^{2x} dx$$

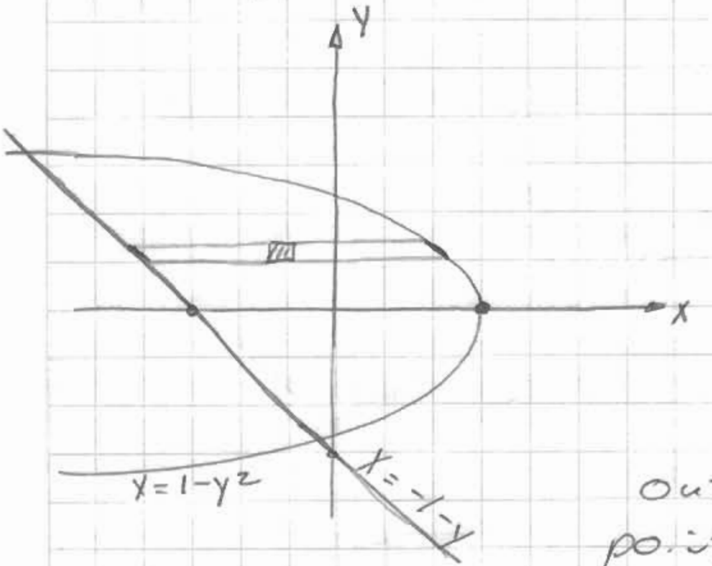
$$= \int_{-1}^{1}\frac{4x^3}{2}+\frac{16x^7}{4}-\frac{x^3}{2}-\frac{x^7}{4}\,dx = \int_{-1}^{1}\frac{3x^3}{2}+\frac{15x^7}{4}\,dx$$

$$= \frac{3x^4}{8} + \frac{15}{32}x^8 \Big|_{-1}^{1} = \frac{3}{8} + \frac{15}{32} - \left(\frac{3}{8} + \frac{15}{32}\right) = 0$$

2 a) $\iint_R xy^2 \, dA$

R: $x + y + 1 = 0$ $\quad x = -1 - y$ (line)

$x + y^2 = 1$ $\quad x = 1 - y^2$ (parabola)



because $x = 1 - y^2$ is familiar shape, ie parabola, integrate in x direction first.

inner integral

$$-1 - y \le x \le 1 - y^2$$

outer integral - find intersection points of line and parabola

$$x = -1 - y = 1 - y^2 \qquad y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$

$$\therefore y = 2, -1$$

$$\iint_R xy^2 \, dA = \int_{-1}^{2} \int_{-1-y}^{1-y^2} xy^2 \, dx \, dy$$

$$= \int_{-1}^{2} \frac{x^2 y^2}{2} \Big|_{-1-y}^{1-y^2} dy = \frac{1}{2} \int_{-1}^{2} (1-y^2)^2 y^2 - (-1-y)^2 y^2 \, dy$$

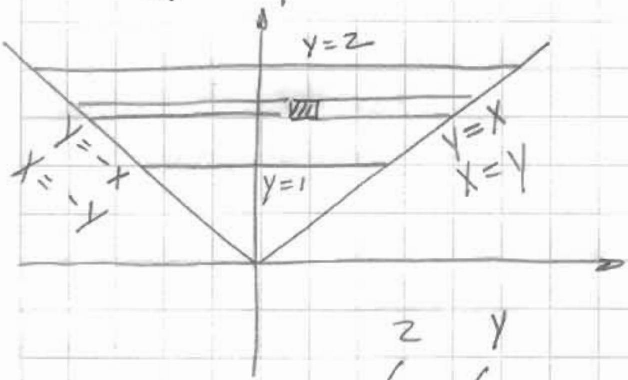
$$= \frac{1}{2} \int_{-1}^{2} \cancel{y^2} - 2y^4 + y^6 - \cancel{y^2} - 2y^3 - y^4 \, dy$$

$$= \frac{1}{2} \int_{-1}^{2} y^6 - 3y^4 - 2y^3 \, dy = \frac{1}{2} \left[\frac{y^7}{7} - \frac{3y^5}{5} - \frac{y^4}{2} \Big|_{-1}^{2} \right]$$

$$= \frac{1}{2} \left[\frac{128}{7} - \frac{96}{5} - \frac{16}{2} - \left(\frac{-1}{7} + \frac{3}{5} - \frac{1}{2} \right) \right]$$

$$= \frac{1}{2} \left[\frac{-621}{70} \right] = \frac{-621}{140} \cong 4.436.$$

2 b) $\iint_R (xy + y^2 - 3x^2)\,dA$ R: $y = |x|$ $y = 1$ $y = 2$



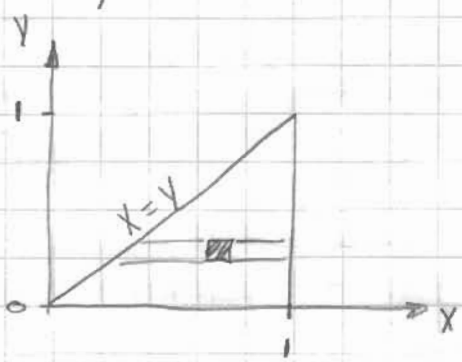
since $y=1$, $y=2$ constant, use them as limits for outer integral in y direction

inner integral $-y \le x \le y$

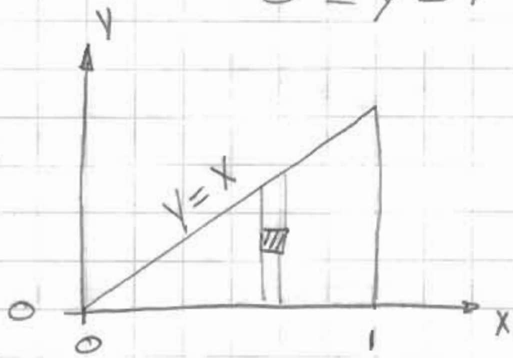
$$= \int_1^2 \int_{-y}^{y} xy + y^2 - 3x^2\,dx\,dy = \int_1^2 \frac{x^2 y}{2} + xy^2 - x^3 \Big|_{-y}^{y} dy$$

$$= \int_1^2 \frac{y^3}{2} + y^3 - y^3 - \left(\frac{y^3}{2} - y^3 + y^3\right) dy = \int_1^2 0\,dy = 0$$

3 a) $\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy$ sketch region $y \le x \le 1$, $0 \le y \le 1$



$\Rightarrow$ reverse order



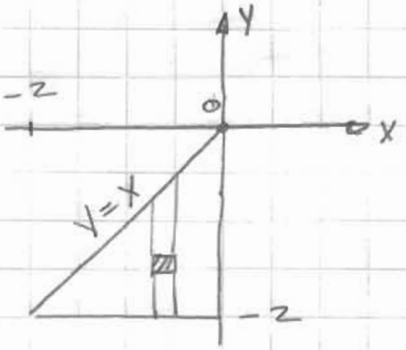
$$\int_0^1 \int_0^x \sin(x^2)\,dy\,dx = \int_0^1 y \cdot \sin(x^2) \Big|_0^x dx = \int_0^1 x \sin(x^2)\,dx$$

substitution $u = x^2$
$du = 2x\,dx$
$x\,dx = \frac{du}{2}$

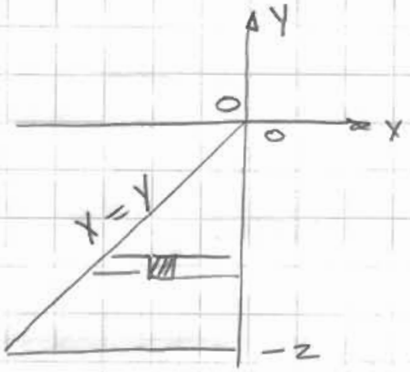
$\int x \sin(x^2)\,dx = \frac{1}{2} \int \sin u\,du$
$= -\frac{1}{2} \cos u$

$$\therefore = \int_0^1 x \sin(x^2)\,dx = -\frac{1}{2} \cos(x^2) \Big|_0^1 = -\frac{1}{2}(\cos 1 - 1)$$

3 b) $\displaystyle\int_{-2}^{0}\int_{-2}^{x} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ sketch region $-2 \le x \le 0$, $-2 \le y \le x$ ⑥



$\Rightarrow$ reverse order



$$\int_{-2}^{0}\int_{y}^{0} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

substitution $u = x^2+y^2$
$du = 2x\,dx$
$x\,dx = \frac{du}{2}$

$$\frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\,\frac{\sqrt{u}}{(1/2)} = \sqrt{u} = \sqrt{x^2+y^2}$$

$$\int_{-2}^{0} \sqrt{x^2+y^2}\,\Big|_{y}^{0} dy = \int_{-2}^{0} \sqrt{y^2} - \sqrt{2y^2}\,dy$$

$$= \int_{-2}^{0} \underbrace{(1-\sqrt{2})\sqrt{y^2}}\,dy$$

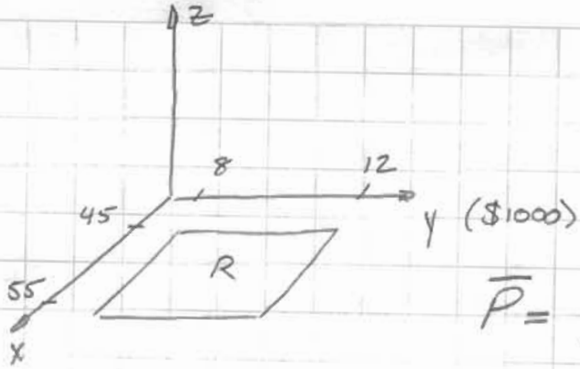
since this term must be positive from $\sqrt{x^2+y^2}$ $\therefore$ evaluate $\sqrt{y^2}$ as $-y$

$$= (\sqrt{2}-1)\int_{-2}^{0} y\,dy = (\sqrt{2}-1)\,\frac{y^2}{2}\Big|_{-2}^{0} = 2(1-\sqrt{2})$$

4 find P_{avg} for $P = 10{,}000\,x^{0.3}\,y^{0.7}$

$45 \le x \le 55 \qquad 8 \le y \le 12$

$$P_{avg} = \frac{1}{A}\iint_{A} P(x,y)\,dA$$



$\therefore \; A = (55-45)\cdot(12-8)$

$= 10 \times 4 = 40$

$$\bar{P} = \frac{1}{40}\cdot \int_{45}^{55}\int_{8}^{12} 10000\, x^{0.3} y^{0.7}\, dy\, dx$$

$$\bar{P} = 250 \int_{45}^{55} x^{0.3} \left.\frac{y^{1.7}}{1.7}\right|_{y=8}^{y=12} dx$$

$$= 250 \left(\left.\frac{x^{1.3}}{1.3}\right|_{45}^{55}\right)\left(\left.\frac{y^{1.7}}{1.7}\right|_{8}^{12}\right)$$

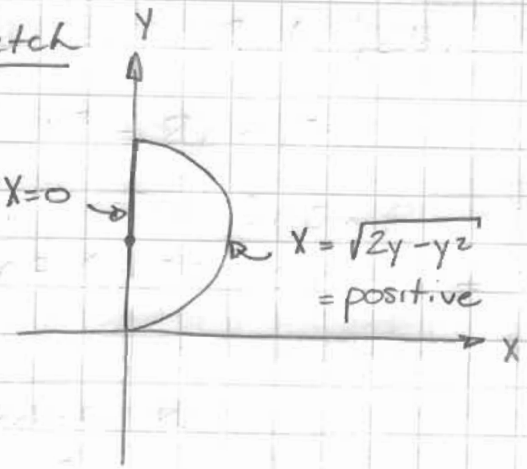
$$= 113.122\,(55^{1.3} - 45^{1.3})(12^{1.7} - 8^{1.7}) = 161{,}781$$

5 a) $\iint_R x\, dA$ R: $x = \sqrt{2y - y^2} \Rightarrow x^2 + y^2 - 2y = 0$

$x = 0$ $\qquad x^2 + (y-1)^2 = 1$

circle with center (0,1)

<u>sketch</u>



convert to polar coordinates

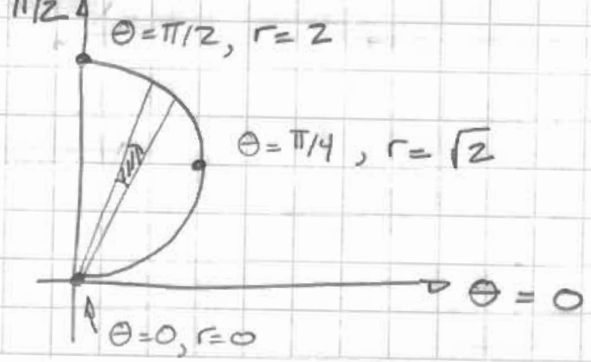
$x = r\cos\theta \qquad y = r\sin\theta$

$x^2 + y^2 - 2y = 0$

$r^2(\cos^2\theta + \sin^2\theta) - 2r\sin\theta = 0$

$r = 2\sin\theta$

sketch in polar coordinates



inner integral

$0 \le r \le 2\sin\theta$

outer integral

$0 \le \theta \le \pi/2$

$$\iint_R x\,dA = \int_0^{\pi/2}\int_0^{2\sin\theta} r\cos\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}\cos\theta \Big|_0^{2\sin\theta} d\theta = \frac{8}{3}\int_0^{\pi/2} \sin^3\theta\cos\theta\,d\theta$$

substitution $u = \sin\theta$, $du = \cos\theta\,d\theta$ $\quad \int u^3\,du = \frac{u^4}{4} = \frac{\sin^4\theta}{4}$

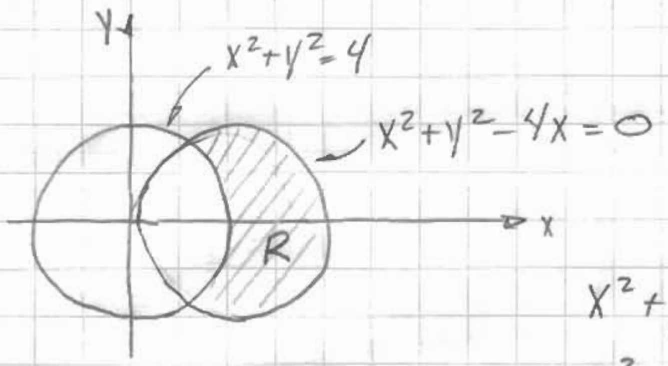
$$= \frac{8}{3}\left[\frac{\sin^4\theta}{4}\right]\Big|_0^{\pi/2} = \frac{2}{3}(1)^4 = \frac{2}{3}$$

5 b) $\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dA$ $\quad$ R: outside $x^2+y^2=4$ ← circle, radius 2

inside $x^2-4x+y^2=0$

$(x-2)^2+y^2=4$ ← circle, radius 2 at $(2,0)$

sketch



convert to polar coordinates

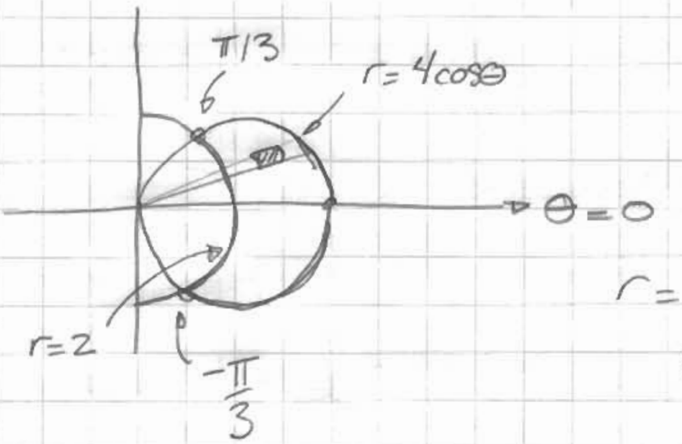
$x^2+y^2=4 \Rightarrow r^2=4$

$r=2$

$x^2+y^2=4x$

$r^2=4r\cos\theta \qquad r=4\cos\theta$

sketch in polar coordinates



∴ inner integral

$2 \le r \le 4\cos\theta$

solve for intersection points

$r = 2 = 4\cos\theta$

$\cos\theta = \frac{1}{2} \qquad \theta = -\frac{\pi}{3}, \frac{\pi}{3}$

∴ Outer integral $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$

$$\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dA = \int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta} \frac{1}{r}\, r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} r\Big|_2^{4\cos\theta} d\theta = \int_{-\pi/3}^{\pi/3} 4\cos\theta - 2\,d\theta$$

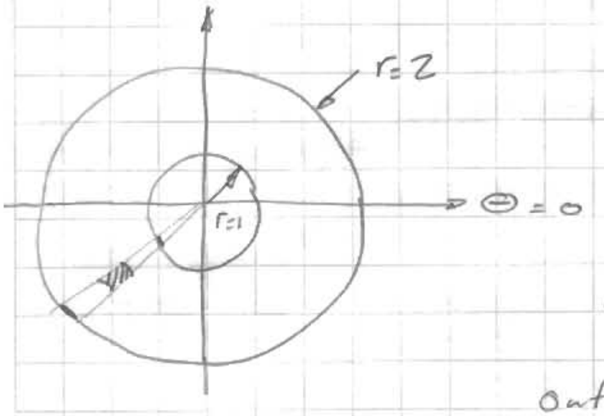
$$= 4\sin\theta - 2\theta\Big|_{-\pi/3}^{\pi/3} = 4\sin\left(\frac{\pi}{3}\right) - \frac{2\pi}{3} - \left(4\sin\left(\frac{-\pi}{3}\right) + \frac{2\pi}{3}\right)$$

$$= 4\left(\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\right) - \frac{4\pi}{3} = 4\sqrt{3} - \frac{4\pi}{3}$$

5 c)

$$\iint_R \sqrt{1+2x^2+2y^2}\,dA \qquad R: \left.\begin{matrix} x^2+y^2=1 \\ x^2+y^2=4 \end{matrix}\right\} \text{ circles, radii 1 and 2}$$

Sketch in polar coordinates



convert to polar form

$x = r\cos\theta$ $\quad x^2+y^2=1 \Rightarrow r=1$

$y = r\sin\theta$ $\quad x^2+y^2=4 \Rightarrow r=2$

$\therefore$ inner integral $1 \le r \le 2$

outer integral $0 \le \theta \le 2\pi$

$$\iint_R \sqrt{1+2x^2+2y^2}\,dA = \int_0^{2\pi}\int_1^2 \sqrt{1+2r^2}\; r\,dr\,d\theta$$

substitute $u = 1+2r^2$ $\quad \int \frac{\sqrt{u}\,du}{4} = \frac{1}{4}\cdot\frac{2}{3}\cdot u^{3/2}$

$du = 4r\,dr$

$r\,dr = \frac{du}{4}$

$$= \frac{1}{6}\int_0^{2\pi} (1+2r^2)^{3/2}\Big|_1^2 d\theta = \frac{2\pi}{6}\left[(1+8)^{3/2} - (1+2)^{3/2}\right] = \pi(9-\sqrt{3})$$

6

$$v = \frac{P}{4nL}(R^2 - r^2)$$

volume flow rate $\dot{V} = \iint_R v \, dA$

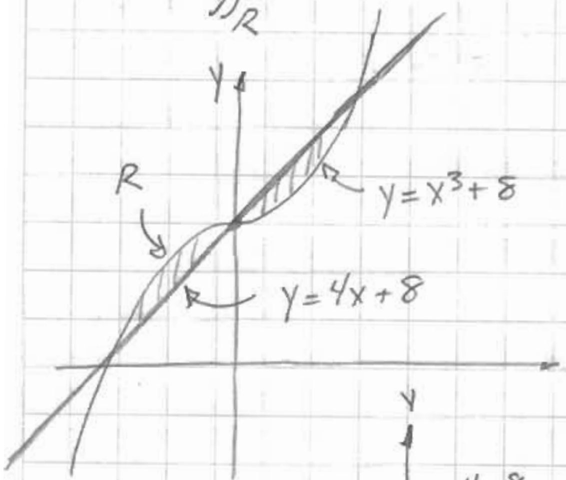
perform integral in polar coordinates. $0 \le r \le R$, $0 \le \theta \le 2\pi$

$$\dot{V} = \int_0^{2\pi}\int_0^R \frac{P}{4nL}(R^2 - r^2)\, r\, dr\, d\theta$$

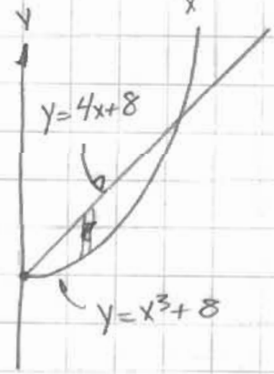
$$= \frac{P}{4nL}(2\pi)\int_0^R R^2 r - r^3 \, dr$$

$$= \frac{P\pi}{2nL}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]\Bigg|_0^R = \frac{P\pi}{2nL}\cdot\frac{R^4}{4} = \frac{P\pi R^4}{8nL}$$

7 a) $A = \iint_R dA$ for R: $y = x^3 + 8$, $y = 4x + 8$



$\Rightarrow$ since functions are symmetric about $x = 0$, use two times area in first quadrant $(x > 0)$



because limits given as $f(x)$, integrate in y direction first

$\therefore$ inner integral

$$x^3 + 8 \le y \le 4x + 8$$

find intersection points.

$y = x^3 + 8 = 4x + 8$ $\quad x = 0$ or $x^2 = 4$, $x = \pm 2$

$\therefore$ outer integral $0 \le x \le 2$

(11)

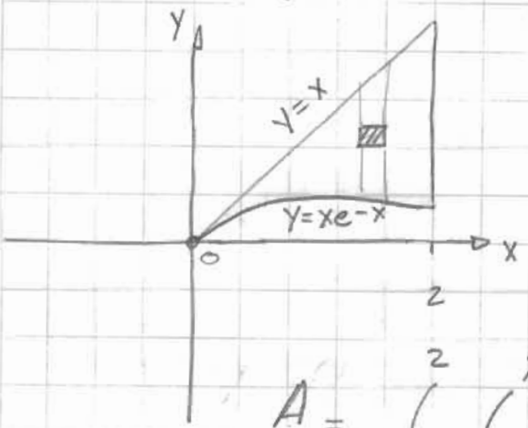
$$A = 2\int_0^2 \int_{x^3+8}^{4x+8} dy\,dx = 2\int_0^2 y \Big|_{x^3+8}^{4x+8} dx$$

$$= 2\int_0^2 4x+8-x^3-8\,dx = 2\left[\frac{4x^2}{2} - \frac{x^4}{4}\right]_0^2$$

$$= 2(8-4) = 8$$

7 b) $A = \iint_R dA \qquad R: \; y = xe^{-x}$

$y = x \qquad x = 2$



$\therefore$ inner integral

$$xe^{-x} \le y \le x$$

outer integral $\quad 0 \le x \le 2$

$$A = \int_0^2 \int_{xe^{-x}}^{x} dy\,dx = \int_0^2 x - xe^{-x}\,dx$$

Schaums. $\displaystyle\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$ let $a = -1$

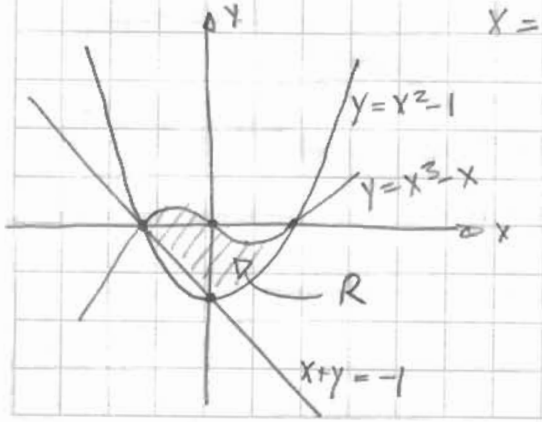
$$A = \frac{x^2}{2}\Big|_0^2 - \left(\frac{e^{-x}}{(-1)}\left(x - \frac{1}{(-1)}\right)\right)\Big|_0^2 = 2 + e^{-x}(x+1)\Big|_0^2$$

$$= 2 + 3e^{-2} - 1 = 1 + 3e^{-2}$$

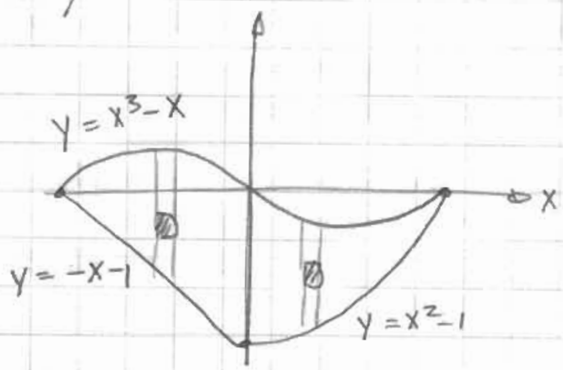
7 c) $\iint_R dA \qquad R: \; y = x^3 - x$

$x + y + 1 = 0 \qquad x + y = -1$

$x = \sqrt{y+1} \quad \Rightarrow \quad y = x^2 - 1$



$\Longrightarrow$



⇒ since boundaries are given as $f(x)$, integrate in y direction first -

⇒ since lower bound on y changes at $x=0$, perform integration in two parts

$$A = \int_{-1}^{0}\int_{-x-1}^{x^3-x} dy\,dx + \int_{0}^{1}\int_{x^2-1}^{x^3-x} dy\,dx$$

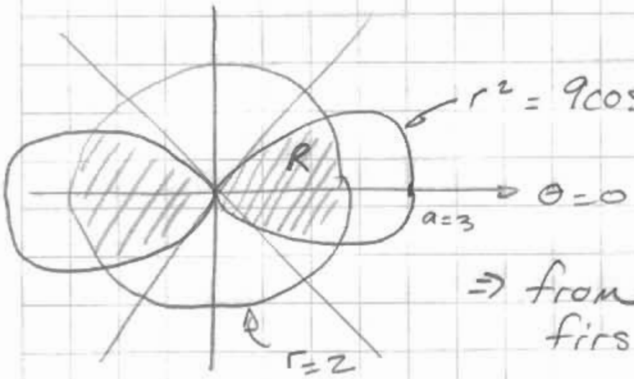
$$= \int_{-1}^{0} x^3 - x + x + 1 \, dx + \int_{0}^{1} x^3 - x - x^2 + 1 \, dx$$

$$= \frac{x^4}{4} + x \Big|_{-1}^{0} + \frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x \Big|_{0}^{1}$$

$$= -\frac{1}{4} + 1 + \frac{1}{4} - \frac{1}{3} - \frac{1}{2} + 1 = \frac{7}{6}$$

7 d) $A = \iint_R dA$ R: $r = 2$

$r^2 = 9\cos 2\theta$ ← Lemniscate (Schaums, pg 25)

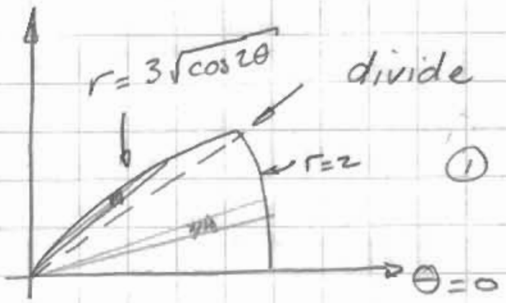


$r^2 = a^2 \cos 2\theta$

$a = 3$

⇒ from symmetry - $A = 4 \times$ area in first quadrant,

∴ outer integral $0 \le \theta \le \pi/4$



divide inner integral into two parts

① $0 \le r \le 2$

② $0 \le r \le 9\cos 2\theta$

find intersection point

$r = 2 = 3\sqrt{\cos 2\theta}$ $\cos 2\theta = \left(\frac{2}{3}\right)^2$ $\theta = 0.555$

(13)

$$A = \int_0^{0.555}\int_0^{2} r\,dr\,d\theta + \int_{0.555}^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

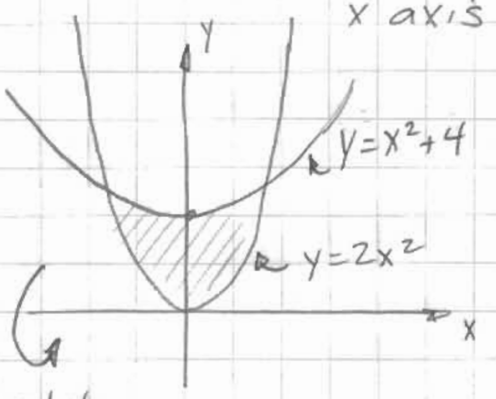
$$= \int_0^{0.555} \frac{r^2}{2}\Big|_0^2 d\theta + \int_{0.555}^{\pi/4} \frac{r^2}{2}\Big|_0^{3\sqrt{\cos 2\theta}} d\theta$$

$$= 2(0.555) + \frac{9}{2}\int_{0.555}^{\pi/4} \cos 2\theta\, d\theta = 2(0.555) + \frac{9}{4}\sin 2\theta \Big|_{0.555}^{\pi/4}$$

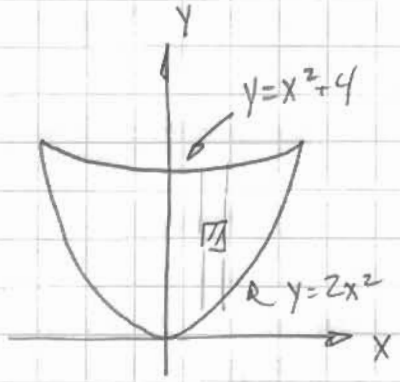
$$= 1.3447$$

$\therefore$ total area $= (4)(1.3447) = 5.3787$

8 a) $V = \iint_R \underbrace{2\pi y}_{\text{rotation about } x \text{ axis, } y=0}\, dA$ $\qquad R: \; y = x^2+4, \; y = 2x^2$



$\Rightarrow$



$\therefore$ inner integral

$2x^2 \le y \le x^2+4$

find intersection

$y = 2x^2 = x^2+4$

$x^2 = 4 \quad x = \pm 2$

$\therefore$ outer integral

$-2 \le x \le 2$

$$V = \int_{-2}^{2}\int_{2x^2}^{x^2+4} 2\pi y\, dy\, dx = \pi\int_{-2}^{2} y^2\Big|_{2x^2}^{x^2+4} dx$$

$$= \pi\int_{-2}^{2} x^4 + 8x^2 + 16 - 4x^4\, dx = \pi\left(-\frac{3x^5}{5} + \frac{8}{3}x^3 + 16x\right)\Big|_{-2}^{2}$$

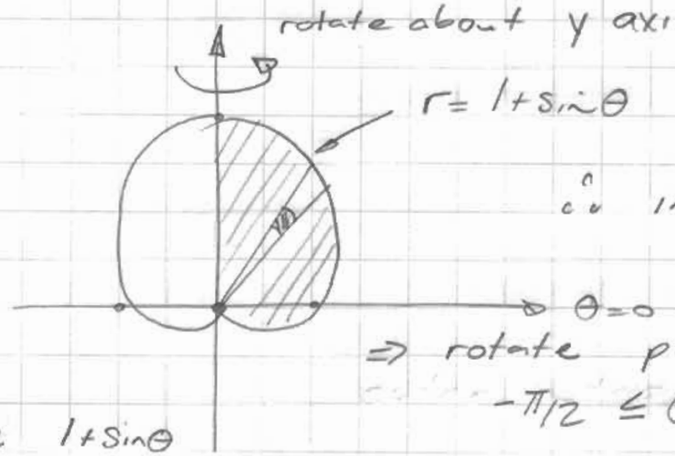
$$= \pi\left(-\frac{3}{5}(32) + \frac{8}{3}(8) + 32 - \left(\frac{3(32)}{5} - \frac{8(8)}{3} - 32\right)\right) = \frac{1024\pi}{15}$$

8 b)

$$V = \iint_R 2\pi x \, dA \qquad R: \; r = 1 + \sin\theta$$

sketch R in polar coordinates

θ	r
0	1
$\pi/2$	2
π	1
$3\pi/2$	0



$r = 1+\sin\theta \Rightarrow$ cardioid

$\therefore$ inner integral

$0 \le r \le 1 + \sin\theta$

$\Rightarrow$ rotate plane to right of y axis

$-\pi/2 \le \theta \le \pi/2$

$$V = \int_{-\pi/2}^{\pi/2} \int_0^{1+\sin\theta} 2\pi (r\cos\theta)\, r\,dr\,d\theta = 2\pi \int_{-\pi/2}^{\pi/2} \frac{r^3}{3}\cos\theta \Big|_0^{1+\sin\theta} d\theta$$

$$= \frac{2\pi}{3} \int_{-\pi/2}^{\pi/2} (1+\sin\theta)^3 \cos\theta \, d\theta$$

substitution

$u = 1+\sin\theta \qquad \int u^3 du = \frac{u^4}{4}$

$du = \cos\theta \, d\theta$

$$= \frac{2\pi}{3}\left[\frac{(1+\sin\theta)^4}{4} \Big|_{-\pi/2}^{\pi/2} \right] = \frac{2\pi}{12}\left((1+1)^4 - (1-1)^4 \right) = \frac{32\pi}{12} = \frac{8\pi}{3}$$

9 a)

$$S = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \; dA \qquad f(x,y) = \sqrt{2xy}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{2xy}} \cdot 2y = \frac{y}{\sqrt{2xy}}$$

$$\frac{\partial f}{\partial y} = \frac{x}{\sqrt{2xy}}$$

$$\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{\frac{x^2+y^2}{2xy} + 1}$$

$$= \sqrt{\frac{x^2+2xy+y^2}{2xy}} = \frac{(x+y)}{\sqrt{2xy}}$$

$$S = \int_1^2 \int_1^3 \frac{x}{\sqrt{2xy}} + \frac{y}{\sqrt{2xy}} \, dy\,dx = \frac{1}{\sqrt{2}} \int_1^2 \int_1^3 \frac{\sqrt{x}}{\sqrt{y}} + \frac{\sqrt{y}}{\sqrt{x}} \, dy\,dx$$

$$= \frac{1}{\sqrt{2}} \int_1^2 \frac{\sqrt{x}\sqrt{y}}{(1/2)} + \frac{2}{3}\frac{y^{3/2}}{\sqrt{x}} \Big|_1^3 dx$$

$$= \frac{1}{\sqrt{2}} \int_1^2 2\sqrt{3}\sqrt{x} + \frac{2}{3}\,\frac{3\sqrt{3}}{\sqrt{x}} - 2\sqrt{x} - \frac{2}{3}\,\frac{1}{\sqrt{x}}\, dx$$

$$= \sqrt{2} \int_1^2 (\sqrt{3}-1)\sqrt{x} + \left(\sqrt{3}-\frac{1}{3}\right)\frac{1}{\sqrt{x}}\, dx$$

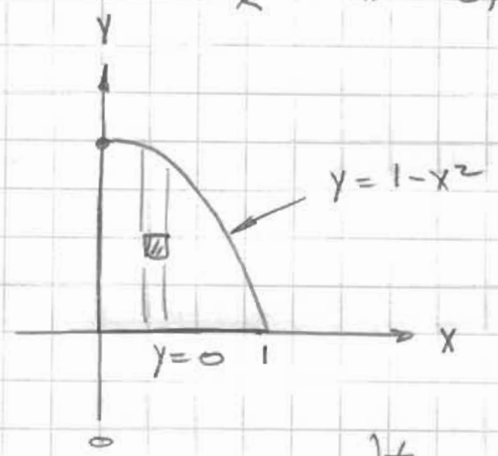
$$= \sqrt{2}\,(\sqrt{3}-1)\cdot\frac{2}{3}\,x^{3/2} + \sqrt{2}\left(\sqrt{3}-\frac{1}{3}\right)\cdot 2x^{\frac{1}{2}}\,\Big|_1^2$$

$$= \frac{2\sqrt{2}}{3}(\sqrt{3}-1)(2\sqrt{2}-1) + 2\sqrt{2}\left(\sqrt{3}-\frac{1}{3}\right)(\sqrt{2}-1)$$

$$= 2.90065$$

9 b) $S = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dA \qquad f = \ln(1+x+y)$

$R = y = 1-x^2$ in 1rst octant



inner integral

$0 \le y \le 1-x^2$

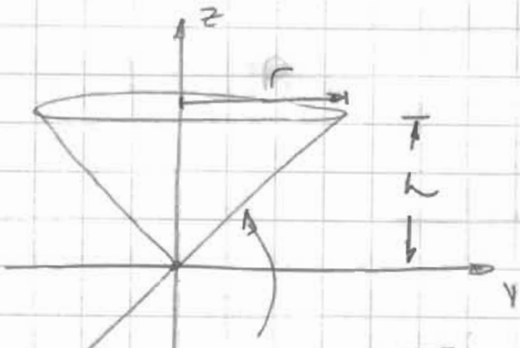
outer integral

$0 \le x \le 1$

$$\frac{\partial f}{\partial x} = \frac{1}{1+x+y} \qquad \frac{\partial f}{\partial y} = \frac{1}{1+x+y}$$

$$S = \int_0^1 \int_0^{1-x^2} \sqrt{\frac{2}{(1+x+y^2)^2} + 1}\; dy\, dx$$

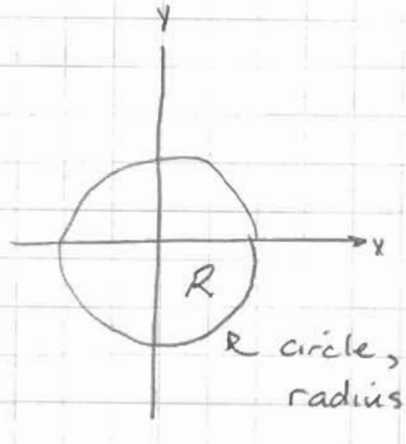
10 a) surface area of cone.



$z = \frac{h}{r}\sqrt{x^2+y^2} \Rightarrow$

$\Rightarrow$ project area on xy plane

Schaum's pg. 36



R circle, radius $= r$

$$\frac{\partial f}{\partial x} = \frac{h}{r}\,\frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial f}{\partial y} = \frac{h}{r}\,\frac{y}{\sqrt{x^2+y^2}}$$

$$S = \iint_R \sqrt{\left(\frac{h}{r}\right)^2 \frac{(x^2+y^2)}{(x^2+y^2)} + 1}\; dA = \iint_R \sqrt{\frac{h^2+r^2}{r^2}}\; dA$$

$$= \frac{\sqrt{h^2+r^2}}{r} \iint_R dA$$

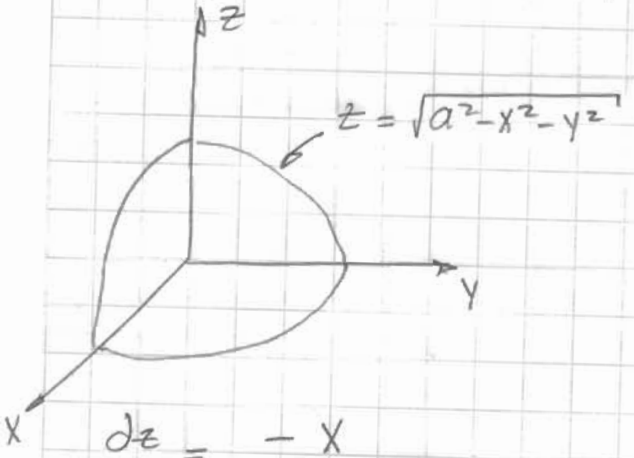
since h, r constant values

equals area of circle on xy plane $= \pi r^2$

$$S = \pi r \sqrt{h^2+r^2}$$

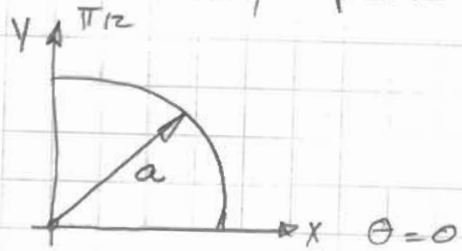
10 b) surface area of sphere.

equation $x^2+y^2+z^2 = a^2$



$\Rightarrow$ consider 1rst octant

$\Rightarrow$ project on xy plane



$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2-x^2-y^2}} \qquad S = \iint_R \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2} + 1}\; dA$$

$$= \iint_R \sqrt{\frac{\cancel{x^2}+\cancel{y^2}+a^2-\cancel{x^2}-\cancel{y^2}}{a^2-x^2-y^2}}\; dA$$

convert to polar coordinates

$$= a \iint_R \frac{1}{\sqrt{a^2-r^2}}\; dA$$

integrate over first quadrant $\quad 0 \le r \le a$, $\quad 0 \le \theta \le \pi/2$

$$S = a \int_0^{\pi/2} \int_0^a \frac{r}{\sqrt{a^2-r^2}} \, dr \, d\theta$$

substitution

$u = a^2 - r^2 \qquad -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u}$

$du = -2r\,dr$

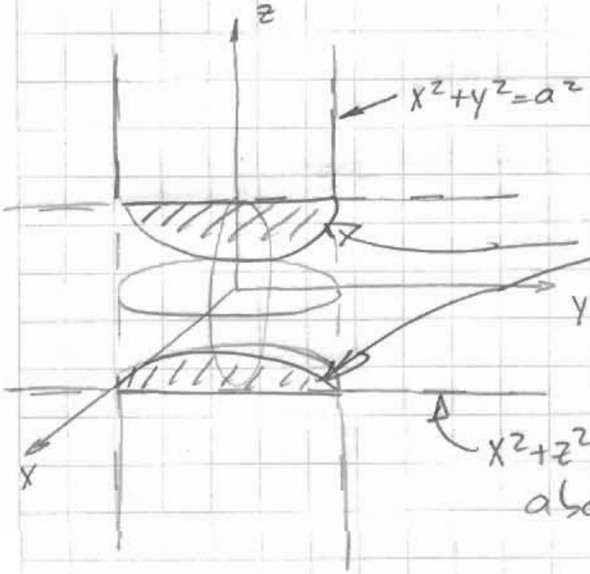
$r\,dr = -\frac{du}{2}$

$$S = a \int_0^{\pi/2} -\sqrt{a^2-r^2}\,\Big|_0^a \, d\theta = a(\pi/2)(-0+a) = \frac{\pi a^2}{2}$$

$\therefore$ total surface area $= 8 \times$ first octant

$= 4\pi a^2$

11) $x^2+z^2=a^2$ inside $x^2+y^2=a^2$



$x^2+y^2=a^2$ = cylinder about z axis

surface of intersection

$x^2+z^2=a^2$ = cylinder about y axis.

from vertical cylinder

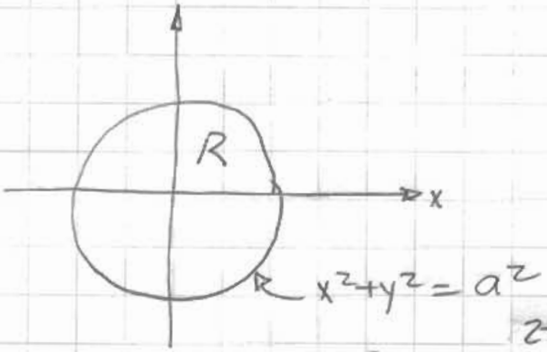
$$S = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \; dA \qquad R: x^2+y^2=a^2$$

$f = z = \sqrt{a^2-x^2}$ from horizontal cylinder

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{a^2-x^2}} \qquad \frac{\partial f}{\partial y} = 0$$

$$\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{\frac{x^2}{a^2-x^2} + \frac{a^2-x^2}{a^2-x^2}} = \frac{a}{\sqrt{a^2-x^2}}$$

sketch R on xy plane



since R is circular, perform integration in polar coordinates

$$0 \le r \le a \qquad 0 \le \theta \le 2\pi$$

$$S = \int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2-r^2\cos^2\theta}}\, r\,dr\,d\theta$$

$$= a\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,dr\,d\theta$$

Substitution

$u = a^2-r^2\cos^2\theta$

$du = -2r\cos^2\theta\,dr$

$rdr = \dfrac{du}{-2\cos^2\theta}$

$$= -\frac{a}{2}\int_0^{2\pi} \frac{2}{\cos^2\theta}\sqrt{a^2-r^2\cos^2\theta}\,\Big|_0^a\, d\theta$$

$$= -a\int_0^{2\pi}\frac{1}{\cos^2\theta}\left(a\sqrt{1-\cos^2\theta} - a\right)d\theta$$

$$= a^2\int_0^{2\pi}\left(\frac{1}{\cos^2\theta} - \frac{\sin\theta}{\cos^2\theta}\right)d\theta$$

Schaums

$\int \frac{dx}{\cos^2 x} = \tan x$

& substitution $u=\cos\theta$
$du = -\sin\theta\,d\theta$

$\int \frac{du}{u^2} = -\frac{1}{u}$

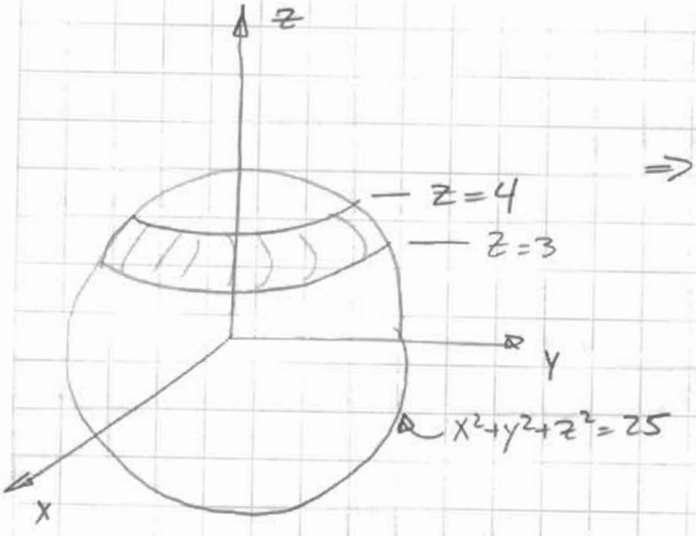
$$= a^2\left(\tan\theta - \frac{1}{\cos\theta}\right)\Big|_0^{2\pi}$$

use symmetry $= 2a^2\left(\tan\theta - \dfrac{1}{\cos\theta}\right)\Big|_0^{\pi}$

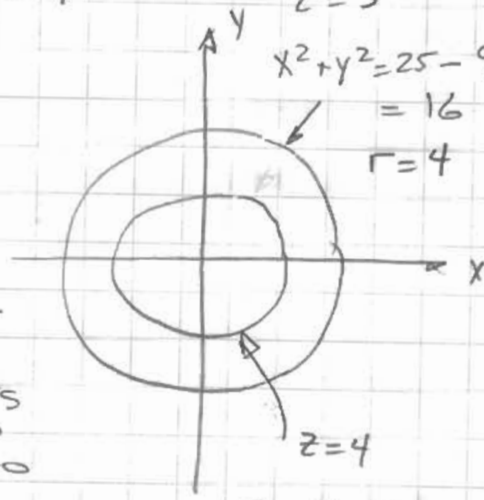
$$= 2a^2\left(-\frac{1}{(-1)} + \frac{1}{(1)}\right) = 4a^2$$

$\therefore$ Counting both surfaces $S = 2(4a^2) = 8a^2$

12 $x^2+y^2+z^2=25$ between $z=3$, $z=4$



$\Rightarrow$ project on xy plane.



$z=3$

$x^2+y^2=25-9$
$=16$
$r=4$

$z=4$

$x^2+y^2=25-16$
$=9$
$r=3$

Since R is circular, convert to polar

$$z=\sqrt{25^2-x^2-y^2}$$

$$\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{25^2-x^2-y^2}}=\frac{-r\cos\theta}{\sqrt{25^2-r^2}}$$

$$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{25^2-x^2-y^2}}=\frac{-r\sin\theta}{\sqrt{25^2-r^2}}$$

$$\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}=\sqrt{\frac{r^2}{25-r^2}+\frac{25-r^2}{25-r^2}}=\frac{5}{\sqrt{25-r^2}}$$

$$S=\int_0^{2\pi}\int_3^4\frac{5}{\sqrt{25-r^2}}\,r\,dr\,d\theta=5\int_0^{2\pi}\int_3^4\frac{r}{\sqrt{25-r^2}}\,dr\,d\theta$$

substitution

$u=25-r^2$

$du=-2r\,dr$

$r\,dr=\frac{du}{-2}$

$-\frac{1}{2}\int\frac{du}{\sqrt{u}}=-\frac{1}{2}\cdot 2\cdot\sqrt{u}$

$=-\sqrt{u}$

$$=5\int_0^{2\pi}-\sqrt{25-r^2}\Big|_3^4\,d\theta$$

$$=10\pi\left(\sqrt{25-9}-\sqrt{25-16}\right)$$

$$=10\pi(4-3)=10\pi.$$