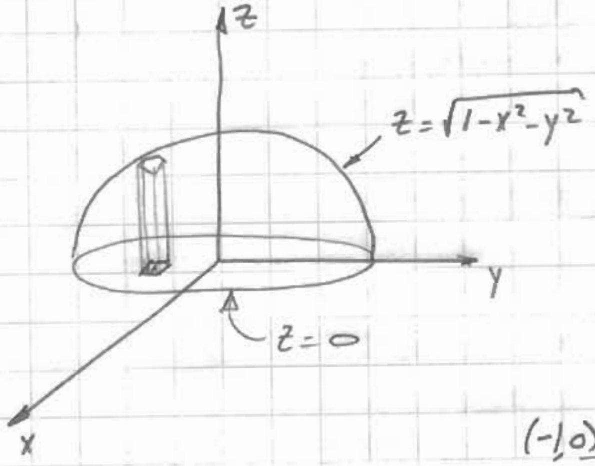
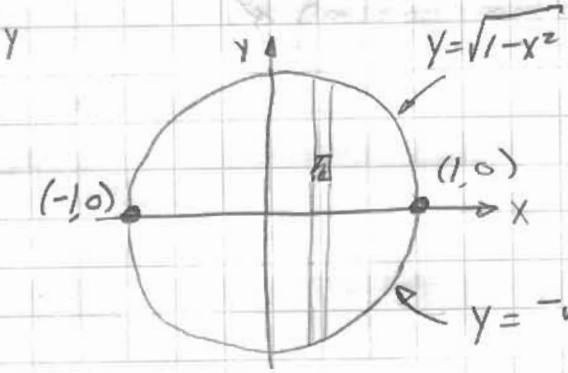
ASSIGNMENT #8 SOLUTIONS

1a) $\iiint_G xy\,dV$ $\qquad$ $G:\ z=\sqrt{1-x^2-y^2} \Rightarrow x^2+y^2+z^2=1$ sphere $z \geq 0$



inner integral

$0 \leq z \leq \sqrt{1-x^2-y^2}$ project on xy plane



$\sqrt{1-x^2-y^2}=0$

$x^2+y^2=1$

$y=\pm\sqrt{1-x^2}$

middle integral $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

outer integral $-1 \leq x \leq 1$

$$\iiint_G xy\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} xy\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\sqrt{1-x^2-y^2}\,dy\,dx$$

substitution

$u=1-x^2-y^2 \quad du=-2y\,dy$

$y\,dy=-\dfrac{du}{2}$

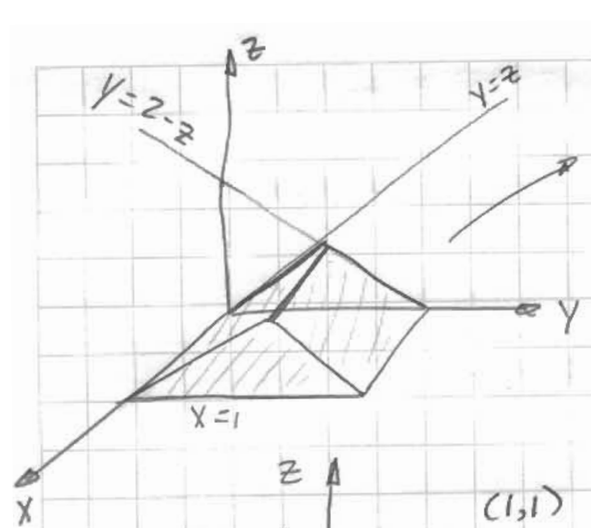
$-\frac{1}{2}\int\sqrt{u}\,du = -\frac{1}{3}u^{3/2}$

$$= \int_{-1}^{1} -\frac{x}{3}(1-x^2-y^2)^{3/2}\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx$$

$$= \int_{-1}^{1} -\frac{x}{3}\left[(1-x^2-1+x^2)-(1-x^2-1+x^2)\right]dx = 0$$

b) $\iiint_G (x+y+z)\,dV$ $\qquad$ $G:\ x=0 \quad y=z \quad z=0$

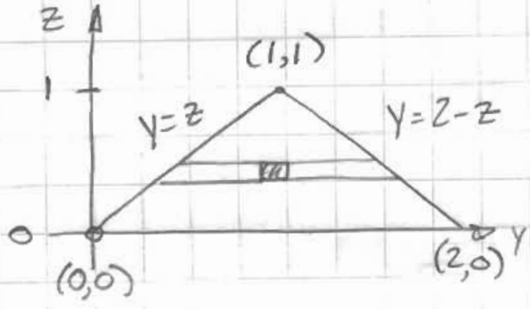
$x=1 \quad y=2-z$



since cross section on yz plane is constant, evaluate outer integral first.

<u>outer integral</u> $0 \le x \le 1$

project on yz plane



<u>inner integral</u>

$z \le y \le 2-z$

<u>middle integral</u>

$0 \le z \le 1$

$$\iiint_G (x+y+z)\,dV = \int_0^1 \int_0^1 \int_z^{2-z} (x+y+z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_0^1 xy + \frac{y^2}{2} + yz \Big|_z^{2-z} dz\,dx$$

$$= \int_0^1 \int_0^1 2x - xz + 2 - 2z + \frac{z^2}{2} + 2z - z^2 - xz - \frac{z^2}{2} - z^2\,dz\,dx$$

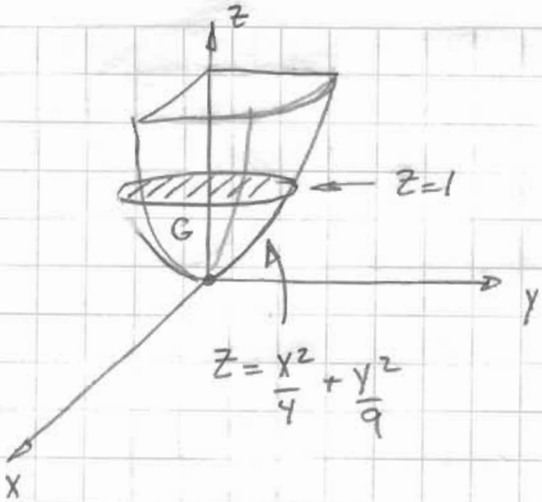
$$= \int_0^1 \int_0^1 2x - 2xz - 2z^2 + 2\,dz\,dx$$

$$= \int_0^1 2xz - xz^2 - \frac{2z^3}{3} + 2z \Big|_0^1 dx = \int_0^1 2x - x - \frac{2}{3} + 2\,dx$$

$$= \int_0^1 x + \frac{4}{3}\,dx = \frac{x^2}{2} + \frac{4}{3}x \Big|_0^1 = \frac{1}{2} + \frac{4}{3} = \frac{11}{6}$$

c) $\iiint_G x^2 y\,dV$ $\quad$ G: $z = \frac{x^2}{4} + \frac{y^2}{9}$, $z = 1$

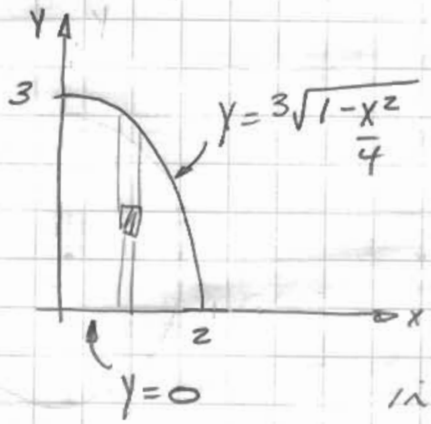
paraboloid



inner integral

$$\frac{x^2}{4} + \frac{y^2}{9} \le z \le 1$$

project G onto xy plane.



$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$y=0,\ x=\pm 2$

$x=0\ \ y=\pm 3$

∴ middle integral

$$0 \le y \le 3\sqrt{1-\frac{x^2}{4}}$$

integrate next in y direction

$$\frac{y^2}{9} = 1 - \frac{x^2}{4} \quad y = \pm 3\sqrt{1-\frac{x^2}{4}}$$

outer integral $\quad 0 \le x \le 2$

$$\iiint_G x^2 y \, dV = \int_0^2 \int_0^{3\sqrt{1-\frac{x^2}{4}}} \int_{\frac{x^2}{4}+\frac{y^2}{9}}^{1} x^2 y \, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{3\sqrt{1-\frac{x^2}{4}}} x^2 y - \frac{x^4 y}{4} - \frac{x^2 y^3}{9} \, dy\, dx$$

$$= \int_0^2 \frac{x^2 y^2}{2} - \frac{x^4 y^2}{8} - \frac{x^2 y^4}{36} \Big|_0^{3\sqrt{1-\frac{x^2}{4}}}$$

$$= \int_0^2 \frac{9}{2} x^2\left(1-\frac{x^2}{4}\right) - \frac{9}{8} x^4\left(1-\frac{x^2}{4}\right) - \frac{81}{36} x^2\left(1-\frac{x^2}{4}\right)^2 dx$$

$$= \frac{9}{2}\int_0^2 x^2 - \frac{x^4}{4} - \frac{x^4}{4} + \frac{x^6}{16} - \frac{9}{18} x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{16}\right) dx$$

$$= \frac{9}{2}\int_0^2 \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{32}\, dx = \frac{9}{2}\left(\frac{x^3}{6} - \frac{x^5}{20} + \frac{x^7}{224}\right)\Big|_0^2$$

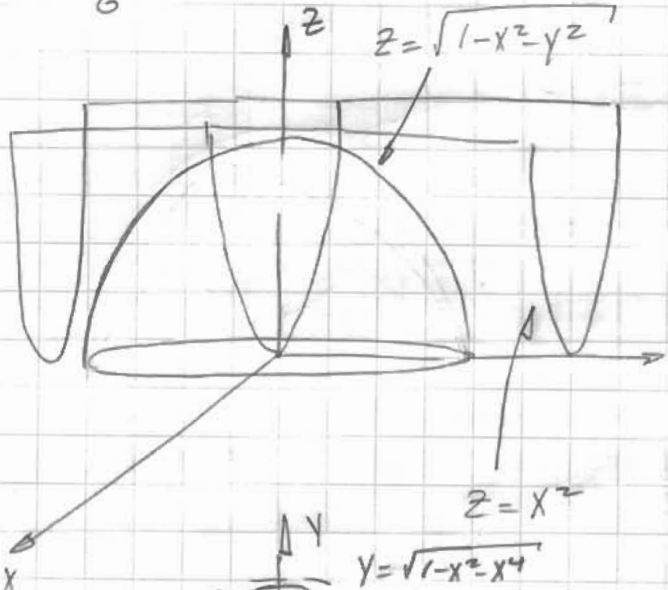
$$= \frac{9}{2}\left(\frac{8}{6} - \frac{32}{20} + \frac{128}{224}\right) = \frac{48}{35}$$

d) $\iiint_G (x^2+y^2+z^2)\,dV$

G: $z = \sqrt{1-x^2-y^2} \Rightarrow x^2+y^2+z^2 = 1$

$\therefore$ sphere, $z \geq 0$

$z = x^2$ – parabola



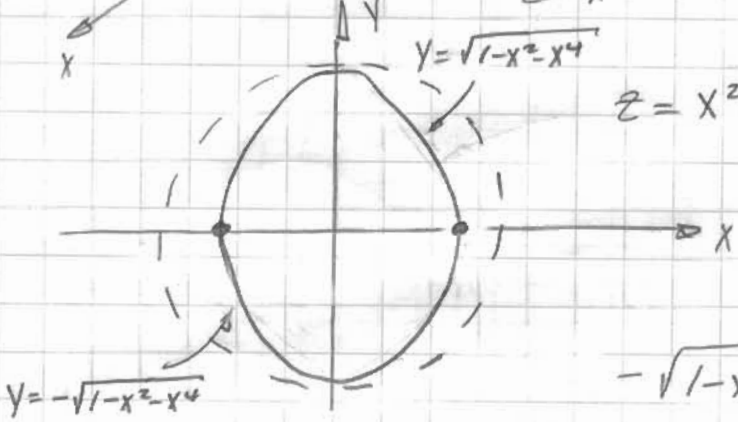
$\therefore$ inner integral

$$x^2 \leq z \leq \sqrt{1-x^2-y^2}$$

project intersection of two surfaces onto xy plane.

$$z = x^2 = \sqrt{1-x^2-y^2} \qquad x^4 = 1-x^2-y^2$$

$$y = \pm\sqrt{1-x^2-x^4}$$



middle integral

$$-\sqrt{1-x^2-x^4} \leq y \leq \sqrt{1-x^2-y^4}$$

find x values when $y = 0 = 1-x^2-x^4$

$$x^4+x^2-1 = 0$$

$$x^2 = \frac{-1 \pm \sqrt{1-4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

$$x = \pm\sqrt{\frac{\sqrt{5}}{2} - \frac{1}{2}}$$

consider only real roots, $+\sqrt{5}$

outer integral

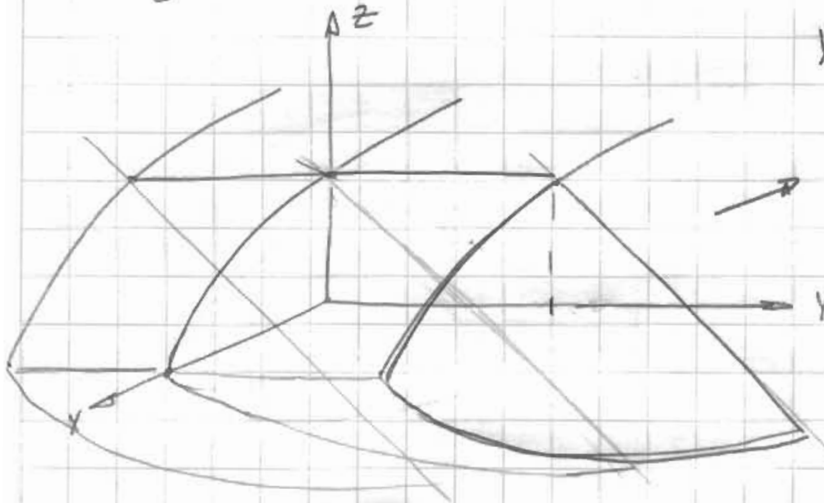
$$-\sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}} \leq x \leq \sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}}$$

$$\iiint_G (x^2+y^2+z^2)\,dV = \int_{-\sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}}}^{\sqrt{\frac{\sqrt{5}}{2}-\frac{1}{2}}} \int_{-\sqrt{1-x^2-x^4}}^{\sqrt{1-x^2-x^4}} \int_{x^2}^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dy\,dx$$

e) $\iiint_G (y+x^2)\,dV$ $\qquad G:\ x+z^2=1 \quad x = 1-z^2$

$z = x+1 \quad x = z-1$

$y=1 \quad y=-1$

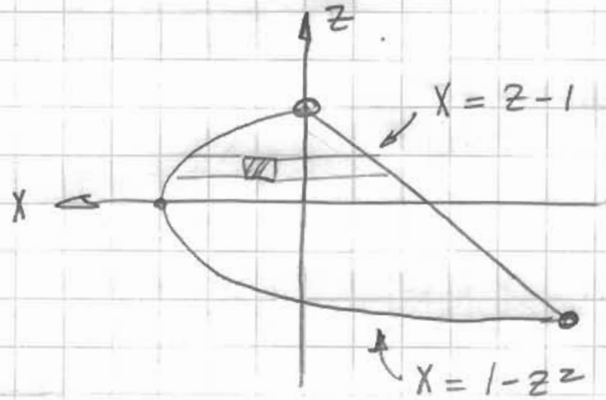


solid has constant c.s. on xz plane

$\therefore$ outer integral

$-1 \le y \le 1$

$\swarrow$ project onto xz plane.



inner integral

$z-1 \le x \le 1-z^2$

find intersection points

$x = z-1 = 1-z^2$

$z^2+z-2=0$

$(z+2)(z-1)=0 \quad z=1,-2$

middle integral $\quad -2 \le z \le 1$

$$\iiint_G (y+x^2)\,dV = \int_{-1}^{1}\int_{-2}^{1}\int_{z-1}^{1-z^2} y+x^2\,dx\,dz\,dy$$

$$= \int_{-1}^{1}\int_{-2}^{1} yx + \frac{x^3}{3}\Big|_{z-1}^{1-z^2} = \int_{-1}^{1}\int_{-2}^{1} y - yz^2 + \frac{(1-z^2)^3}{3} - yz + y - \frac{(z-1)^3}{3}\,dz\,dy$$

$$= \int_{-1}^{1}\int_{-2}^{1} 2y - yz^2 - yz + \frac{2}{3} + z^4 - \frac{z^6}{3} - \frac{z^3}{3} - z\,dz\,dy$$

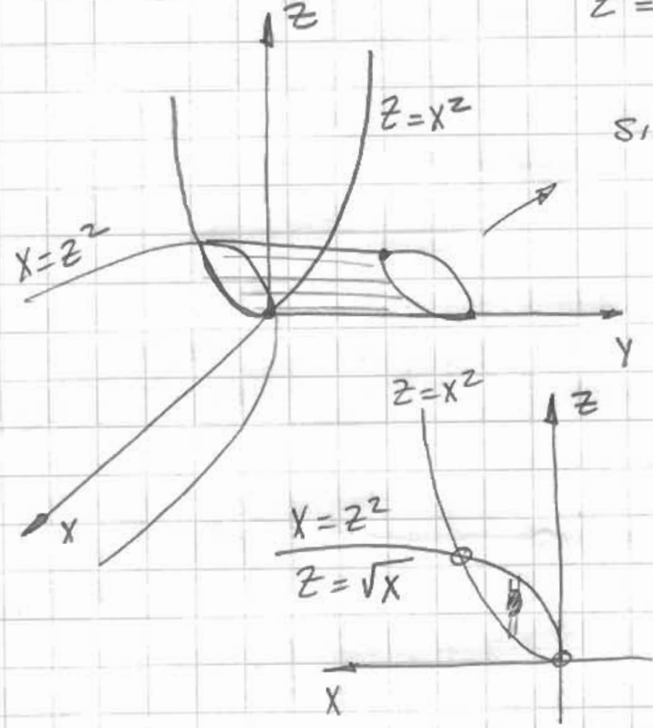
$$= \int_{-1}^{1} 2yz - \frac{yz^3}{3} - \frac{yz^2}{2} + \frac{2}{3}z + \frac{z^5}{5} - \frac{z^7}{21} - \frac{z^4}{12} - \frac{z^2}{2}\Big|_{-2}^{1}\,dy$$

$$= \int_{-1}^{1} \frac{729}{140} + \frac{9}{2} y \, dy = \frac{729 y}{140} + \frac{9y^2}{4} \Big|_{-1}^{1}$$

$$= \frac{729}{140} + \frac{9}{4} + \frac{729}{140} - \frac{9}{4} = \frac{729}{70}$$

2a) $\iiint_G dV$ $\quad$ G: $x = z^2$ $\quad$ $y = 0$

$z = x^2$ $\quad$ $y = 2$



since cross section on xz plane is constant,

<u>outer integral</u>

$0 \le y \le 2$

project onto xz plane.

<u>inner integral</u>

$x^2 \le z \le \sqrt{x}$

intersection points

$z = x^2 = \sqrt{x}$ $\quad$ $x = 0, 1$

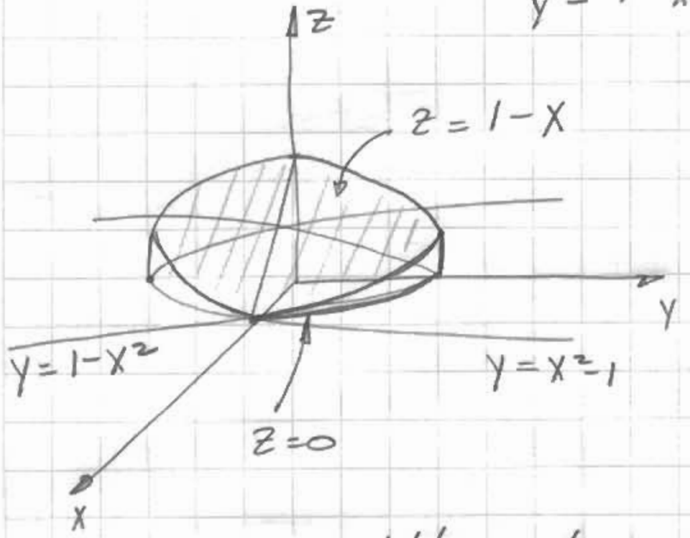
<u>middle integral</u> $\quad 0 \le x \le 1$

$$V = \int_0^2 \int_0^1 \int_{x^2}^{\sqrt{x}} dz\, dx\, dy = \int_0^2 \int_0^1 \sqrt{x} - x^2 \, dx\, dy$$

$$= \int_0^2 \frac{2}{3} x^{3/2} - \frac{x^3}{3} \Big|_0^1 dy = \int_0^2 \frac{2}{3} - \frac{1}{3} \, dy = \int_0^2 \frac{1}{3} dy = \frac{2}{3}$$

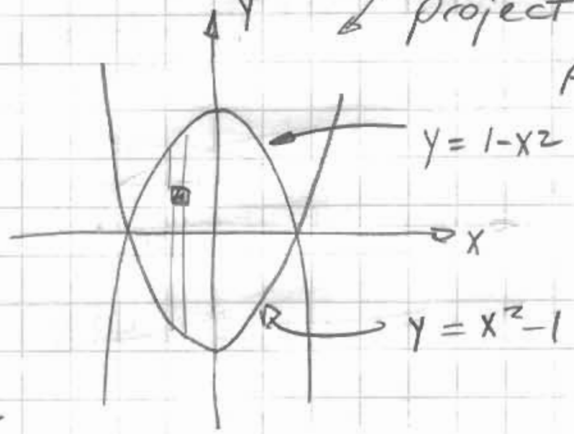
b) $\iiint_G dV \qquad G: \; y = x^2 - 1 \quad x + z = 1$

$\qquad\qquad\qquad\qquad y = 1 - x^2 \quad z = 0$



inner integral $0 \le z \le 1 - x$

project on xy plane



middle integral

$x^2 - 1 \le y \le 1 - x^2$

find intersection points $y = x^2 - 1 = 1 - x^2$

$2x^2 = 2 \quad x^2 = 1 \quad x = \pm 1$

$$V = \int_{-1}^{1} \int_{x^2-1}^{1-x^2} \int_{0}^{1-x} dz\, dy\, dx$$

$$= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} 1 - x \, dy\, dx = \int_{-1}^{1} (1-x)(1 - x^2 - x^2 + 1)\, dx$$

$$= 2\int_{-1}^{1} 1 - x^2 - x + x^3 dx = 2\left(x - \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^4}{4} \right)\Big|_{-1}^{1}$$

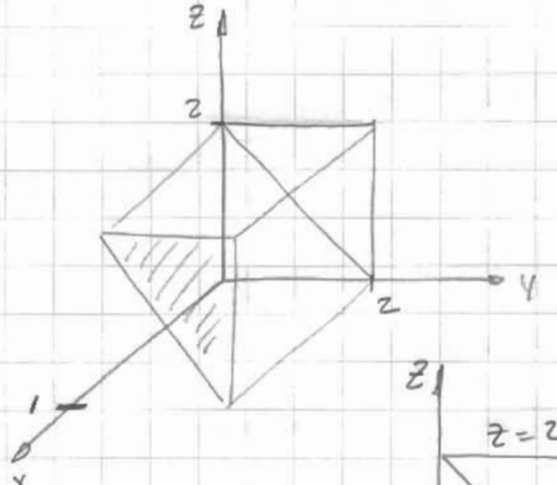
$$= 2\left(1 - \frac{1}{3} - \frac{1}{2} - \frac{1}{4} - \left(-1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{4} \right) \right)$$

$$= 2\left(2 - \frac{2}{3} \right) = 2\left(\frac{4}{3} \right) = \frac{8}{3}$$

3) $\bar{f} = \frac{1}{V} \iiint_G x^2+y^2+z^2 \, dV$ $\quad$ $G: \; x=0, \; x=1$

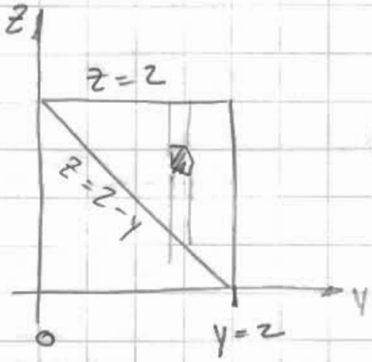
$z = 2-y \qquad y = 2$

$z = 2$



since cs on yz plane constant, define <u>outer integral</u> first

$0 \le x \le 1$ — project on yz plane



<u>inner</u> $\quad 2-y \le z \le 2$

middle $\quad 0 \le y \le 2$

$$\bar{f} = \frac{1}{V}\int_0^1\int_0^2\int_{2-y}^2 x^2+y^2+z^2 \, dz\,dy\,dx = \frac{1}{V}\int_0^1\int_0^2 (x^2+y^2)z + \frac{z^3}{3}\Big|_{2-y}^2 \, dy\,dx$$

$$= \frac{1}{V}\int_0^1\int_0^2 2(x^2+y^2) + \frac{8}{3} - 2(x^2+y^2) + x^2y + y^3 - \frac{(2-y)(4-4y+y^2)}{3} \, dy\,dx$$

$$= \frac{1}{V}\int_0^1\int_0^2 \frac{8}{3} + x^2y + y^3 - \frac{8}{3} + \frac{8y}{3} - \frac{2}{3}y^2 + \frac{4y}{3} - \frac{4y^2}{3} + \frac{y^3}{3} \, dy\,dx$$

$$= \frac{1}{V}\int_0^1\int_0^2 x^2y + 4y - 2y^2 + \frac{4}{3}y^3 \, dy\,dx$$

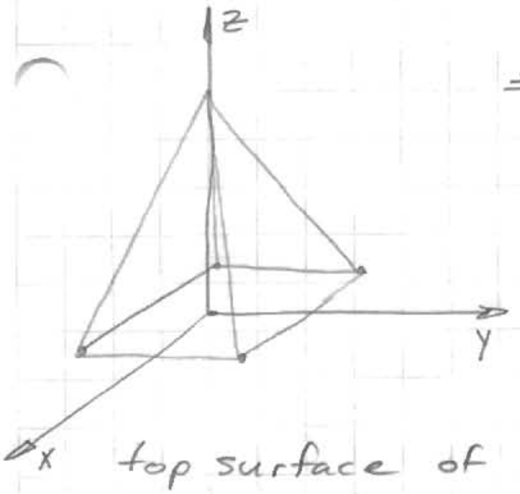
$$= \frac{1}{V}\int_0^1 \frac{x^2y^2}{2} + 2y^2 - \frac{2y^3}{3} + \frac{y^4}{3}\Big|_0^2 \, dx$$

$$= \frac{1}{V}\int_0^1 2x^2 + 8 \, dx = \frac{1}{V}\left(\frac{2x^3}{3} + 8x\right)\Big|_0^1 = \frac{1}{V}\,\frac{26}{3}$$

$V \Rightarrow$ from inspection

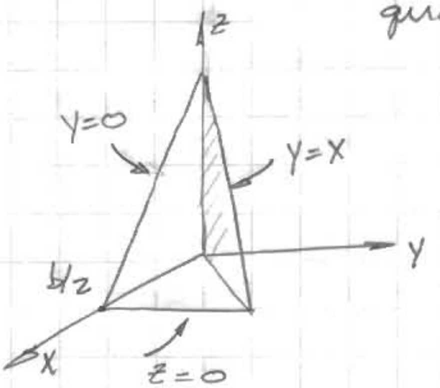
$= (2)(2)(\tfrac{1}{2})(1) = 2$ $\qquad$ $\bar{f} = \frac{13}{3}$

4) pyramid height $= h$
base side length $= b$ $\Rightarrow$ find $\iiint_{\mathcal{V}} d\mathcal{V}$

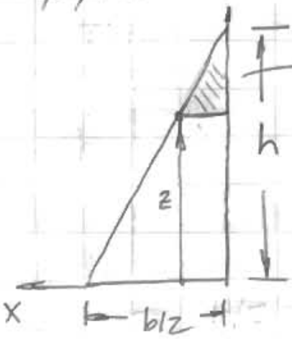


$\Rightarrow$ use symmetry – solve in first quadrant.

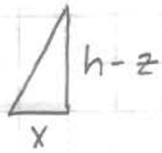
$\Rightarrow$



top surface of pyramid



$\rightarrow$ similar triangles



$\therefore \quad \dfrac{h-z}{x} = \dfrac{h}{b/2}$

$h - z = \dfrac{2xh}{b}$

$z = h - \dfrac{2xh}{b} = h\left(1 - \dfrac{2}{b}x\right)$

$$\mathcal{V} = \iiint_{\mathcal{V}} d\mathcal{V}$$

$$= 8 \int_0^{b/2} \int_0^{x} \int_0^{h(1-\frac{2}{b}x)} dz\, dy\, dx$$

$$= 8h \int_0^{b/2} \int_0^{x} 1 - \frac{2}{b}x \, dy\, dx = 8h \int_0^{b/2} x - \frac{2}{b}x^2 \, dx$$

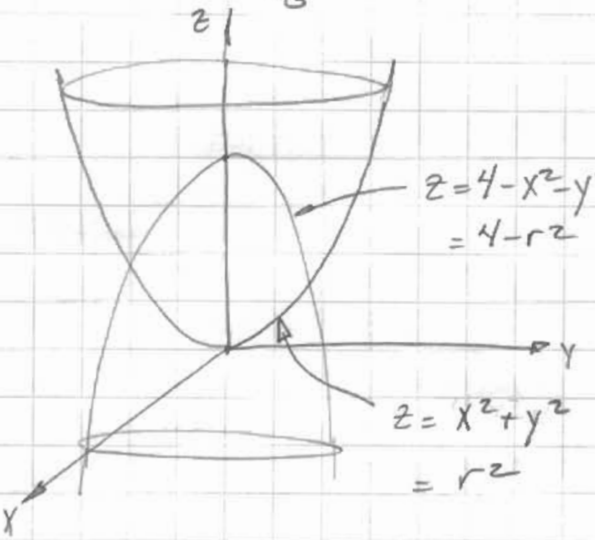
$$= 8h \left[\frac{x^2}{2} - \frac{2}{b}\frac{x^3}{3} \right]\Bigg|_0^{b/2}$$

$$= 8h \left[\frac{b^2}{8} - \frac{2}{3b} \cdot \frac{b^3}{8} \right] = h\left(b^2 - \frac{2}{3}b^2\right)$$

$$= \frac{hb^2}{3}$$

5a) $\mathcal{V} = \iiint_G d\mathcal{V}$ G: $z = x^2+y^2$ ← paraboloid

$z = 4-x^2-y^2$ ← paraboloid



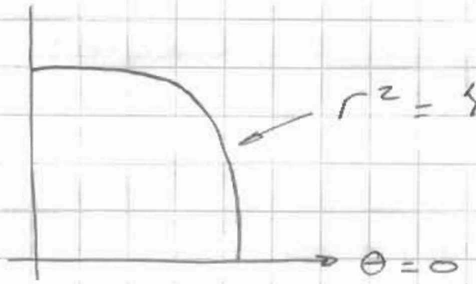
Since solid is symmetric about vertical axis, use polar coordinates

$x = r\cos\theta \quad y = r\sin\theta$

$z = x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$

$z = 4-r^2$

<u>inner integral</u> $r^2 \le z \le 4-r^2$ } project intersection of surfaces onto $r\theta$ plane



$r^2 = 4-r^2 \quad 2r^2 = 4$

$r^2 = 2$

$r = \sqrt{2}$

<u>middle</u> integral $0 \le r \le \sqrt{2}$

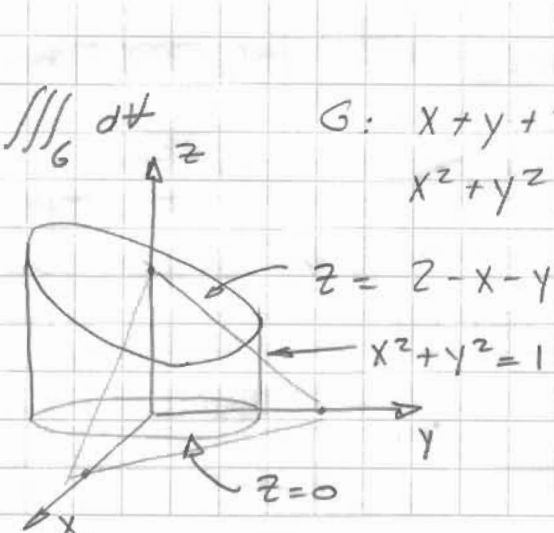
outer integral $0 \le \theta \le 2\pi$

$$\mathcal{V} = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{4-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} 4r - r^3 - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} 2r^2 - \frac{r^4}{2}\Big|_0^{\sqrt{2}} d\theta = 2\pi\left(2\cdot 2 - \frac{4}{2}\right) = 4\pi$$

5b) $\mathcal{V} = \iiint_G d\mathcal{V}$ G: $x+y+z = 2$

$x^2+y^2 = 1 \qquad z = 0$



Since solid is circular about verticle axis, use polar coordinates

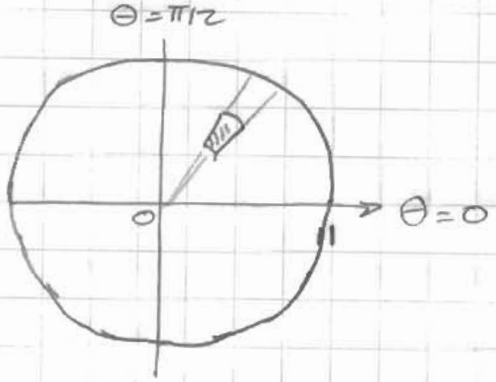
$x = r\cos\theta$ $\qquad z = 2 - r(\cos\theta + \sin\theta)$

$y = r\sin\theta$

$x^2+y^2=1 \Rightarrow r=1$

<u>inner</u> integral $\quad 0 \le z \le 2 - r(\cos\theta + \sin\theta)$

project onto $r\theta$ plane.



middle integral $\quad 0 \le r \le 1$

outer integral $\quad 0 \le \theta \le 2\pi$

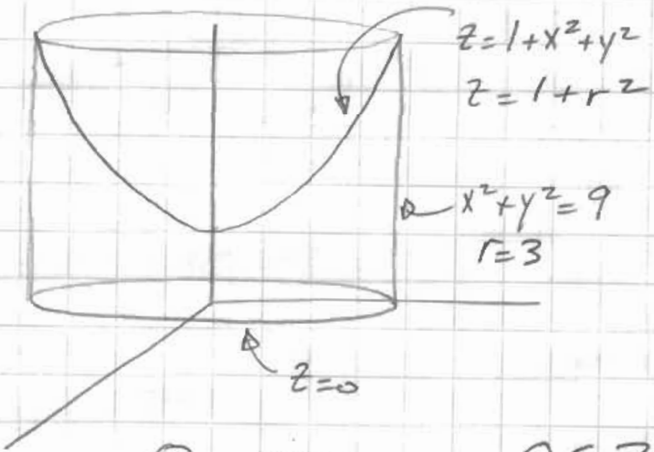
$$V = \int_0^{2\pi}\int_0^1\int_0^{2-r(\cos\theta+\sin\theta)} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r - r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} r^2 - \frac{r^3}{3}(\cos\theta+\sin\theta)\Big|_0^1 d\theta = \int_0^{2\pi} 1 - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\,d\theta$$

$$\left(\theta - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta\right)\Big|_0^{2\pi} = 2\pi + \frac{1}{3} - \frac{1}{3} = 2\pi$$

6) $\iiint_G f(x,y,z)\,dV$ $\qquad G: z = 1 + x^2 + y^2$ ← paraboloid

$x^2+y^2 = 9$, $z = 0$



$x = r\cos\theta \quad y = r\sin\theta$

$z = 1 + x^2 + y^2 \Rightarrow 1 + r^2$

$x^2 + y^2 = 9 \Rightarrow r = 3$

① inner $\quad 0 \le z \le 1 + r^2$

projects circle on $r\theta$ plane

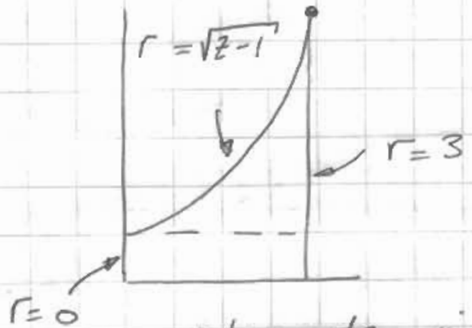
$\therefore$ middle $\quad 0 \le r \le 3$

outer $\quad 0 \le \theta \le 2\pi$

(1) $V = \int_0^{2\pi}\int_0^{3}\int_0^{1+r^2} r\,dz\,dr\,d\theta$

(2) $V = \int_0^{3}\int_0^{2\pi}\int_0^{1+r^2} r\,dz\,d\theta\,dr$ ⇐ switch r, θ integrals

(3) outer $0 \le \theta \le 2\pi$ } projects onto rz plane.



inner $\sqrt{z-1} \le r \le 3$
middle $1 \le z \le 10$

from $\sqrt{z-1} = 3$
$z - 1 = 9$
$z = 10$

inner $0 \le r \le 3$
middle $0 \le z \le 1$

integrate in two parts

(3) $V = \int_0^{2\pi}\int_0^{1}\int_0^{3} r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_1^{10}\int_{\sqrt{z-1}}^{3} r\,dr\,dz\,d\theta$

(4) $V = \int_0^{1}\int_0^{2\pi}\int_0^{3} r\,dr\,dz\,d\theta + \int_1^{10}\int_0^{2\pi}\int_{\sqrt{z-1}}^{3} r\,dr\,dz\,d\theta$

(5) inner $0 \le \theta \le 2\pi$ } projects onto rz plane.

middle $\sqrt{z-1} \le r \le 3$
outer $0 \le z \le 1$

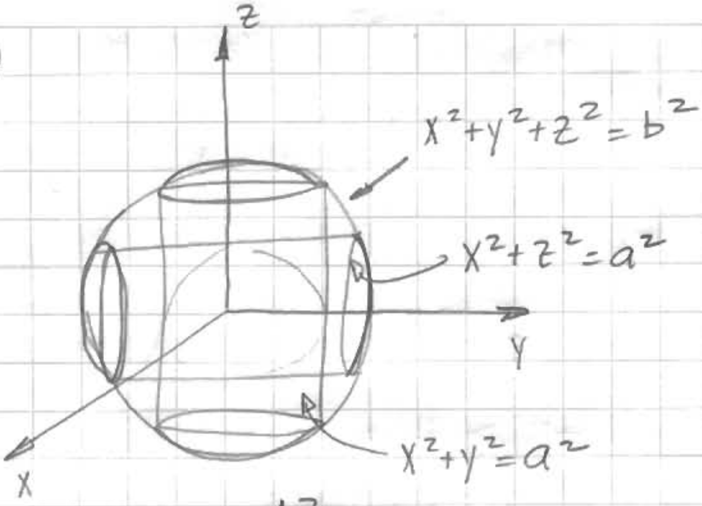
and

middle $0 \le r \le 3$
outer $1 \le z \le 10$

(5) $V = \int_0^{1}\int_{\sqrt{z-1}}^{3}\int_0^{2\pi} r\,d\theta\,dr\,dz + \int_1^{10}\int_0^{3}\int_0^{2\pi} r\,d\theta\,dr\,dz$

(6) $V = \int_0^{3}\int_0^{1+r^2}\int_0^{2\pi} r\,d\theta\,dz\,dr$

7)



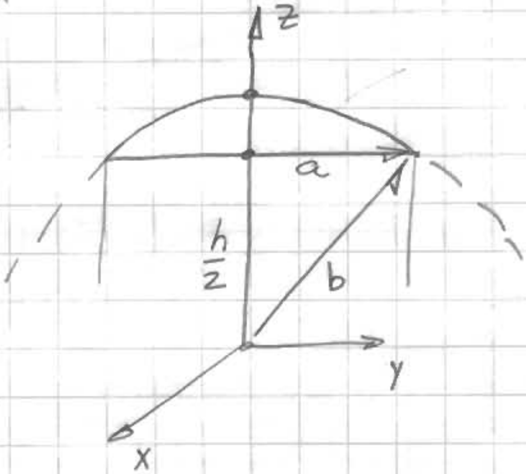
$$V_{total} = V_{sphere} - 2V_{cylinders} - 4V_{caps} + V_{\text{intersection of cylinders}}$$

$$V_{sphere} = \frac{4}{3}\pi b^3$$

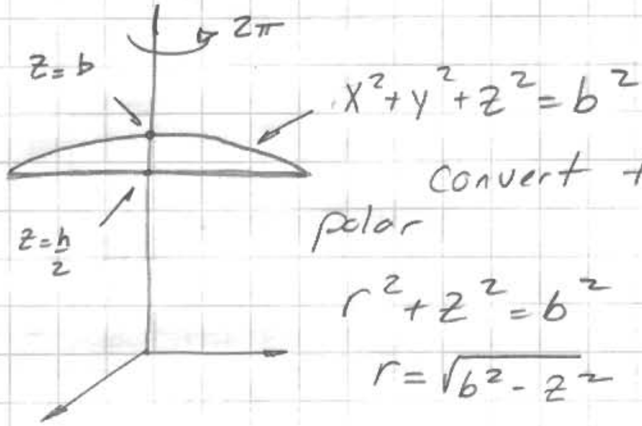
$$V_{cylinders} = \pi a^2 \cdot h$$

$$\frac{h}{2} = \sqrt{b^2-a^2} \quad h = 2\sqrt{b^2-a^2}$$

$$V_{cylinders} = 2\pi a^2\sqrt{b^2-a^2}$$



$\underline{\underline{V_{cap}}}$



convert to polar

$$r^2+z^2=b^2$$

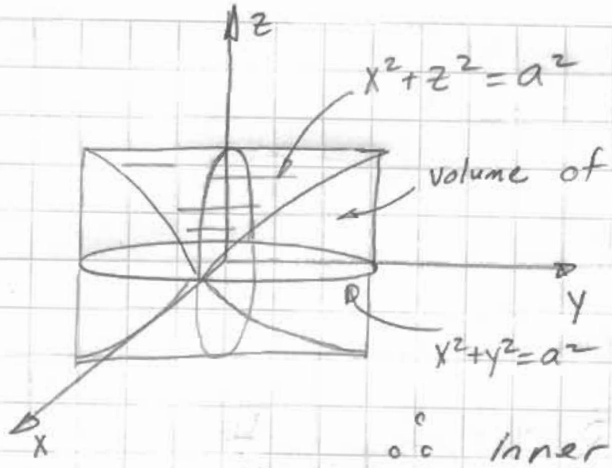
$$r=\sqrt{b^2-z^2}$$

$$V_{cap} = \int_0^{2\pi}\int_{\sqrt{b^2-a^2}}^{b}\int_0^{\sqrt{b^2-z^2}} r\,dr\,dz\,d\theta$$

$$= 2\pi\int_{\sqrt{b^2-a^2}}^{b} \frac{b^2-z^2}{2}\,dz = \pi\left(b^2z-\frac{z^3}{3}\right)\Big|_{\sqrt{b^2-a^2}}^{b}$$

$$= \pi\left(b^3-\frac{b^3}{3} - b^2\sqrt{b^2-a^2} + \frac{(b^2-a^2)\sqrt{b^2-a^2}}{3}\right)$$

$$= \frac{2}{3}\pi b^3 - \pi b^2\sqrt{b^2-a^2} + \frac{\pi}{3}(b^2-a^2)^{3/2}$$

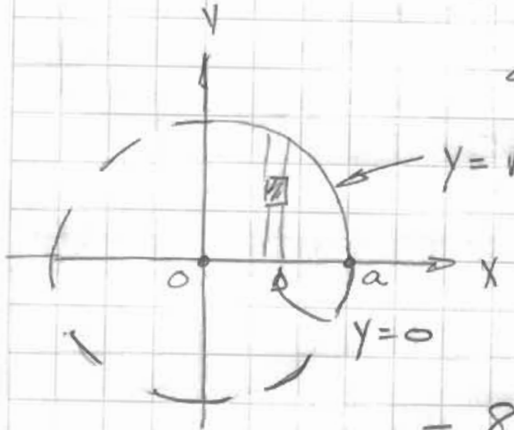


$z = \sqrt{a^2-x^2}$

because of symmetry, consider first octant only

$\therefore$ inner integral $\quad 0 \le z \le \sqrt{a^2-x^2}$

& project intersection on xy plane.



$$\therefore V_{intersection} = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\,dx = 8\int_0^a a^2-x^2\,dx$$

$$= 8\left(a^2x - \frac{x^3}{3}\right)\Big|_0^a = 8\left(a^3 - \frac{a^3}{3}\right) = \frac{16a^3}{3}$$

$$\therefore V_{tot} = \frac{4}{3}\pi b^3 - 2\left(2\pi a^2\sqrt{b^2-a^2}\right) - 4\left(\frac{2}{3}\pi b^3 - \pi b^2\sqrt{b^2-a^2} + \frac{\pi}{3}(b^2-a^2)^{3/2}\right) + \frac{16a^3}{3}$$

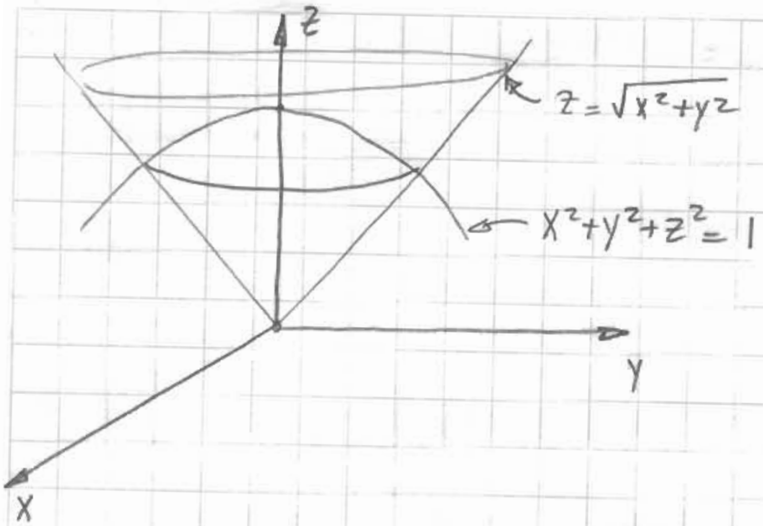
$$= -\frac{4}{3}\pi b^3 + 4\pi(b^2-a^2)\sqrt{b^2-a^2} - \frac{4\pi}{3}(b^2-a^2)^{3/2} + \frac{16a^3}{3}$$

$$= \frac{16a^3}{3} + \frac{4\pi}{3}\left(2(b^2-a^2)^{3/2} - b^3\right)$$

8a) $\iiint_G dV \qquad G: \; z = \sqrt{x^2+y^2} \leftarrow$ cone

$z = \sqrt{1-x^2-y^2} \Rightarrow x^2+y^2+z^2=1 \leftarrow$ sphere

(15)



$x = r\cos\theta\sin\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\phi$

$z = \sqrt{x^2+y^2}$
$r\cos\phi = \sqrt{r^2\sin^2\phi(\cos^2\theta+\sin^2\theta)}$
$r\cos\phi = r\sin\phi$
$\tan\phi = 1 \qquad \phi = \pi/4$

$x^2+y^2+z^2=1$
$r^2\sin^2\phi(\cos^2\theta+\sin^2\theta)+r^2\cos^2\phi = 1$
$r=1$

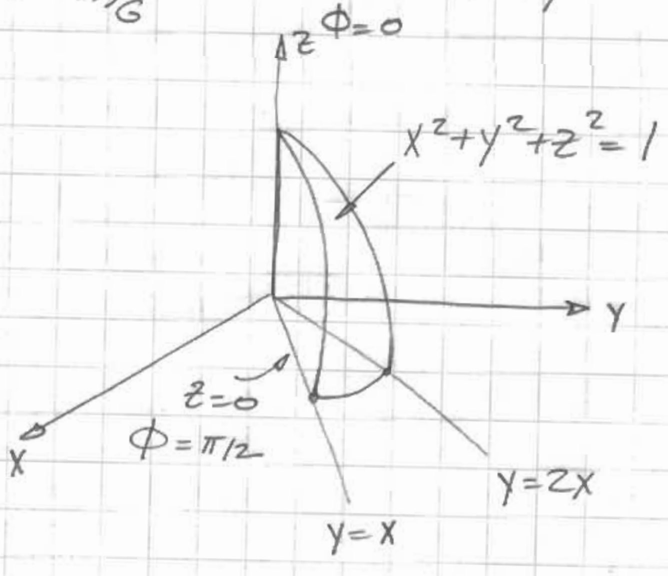
∴ all integrals have constant limits

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 r^2\sin\phi\, dr\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{r^3}{3}\sin\phi \Big|_0^1 d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi} -\cos\phi\Big|_0^{\pi/4} d\theta$$

$$= \frac{1}{3}(2\pi)\left(1-\cos\left(\frac{\pi}{4}\right)\right) = \frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$

b) $\iiint_G dV \qquad G: x^2+y^2+z^2=1 \qquad \begin{array}{l} y=x \\ y=2x \\ z=0 \end{array}$ in first octant



$x^2+y^2+z^2=1 \Rightarrow r=1$

$y=x \Rightarrow r\sin\phi\sin\theta = r\sin\phi\cos\theta$
$\frac{\sin\theta}{\cos\theta}=1 \qquad \theta=\frac{\pi}{4}$

$y=2x \Rightarrow r\sin\phi\sin\theta = 2r\sin\phi\cos\theta$
$\frac{\sin\theta}{\cos\theta}=2 \qquad \theta=\tan^{-1}(2)$

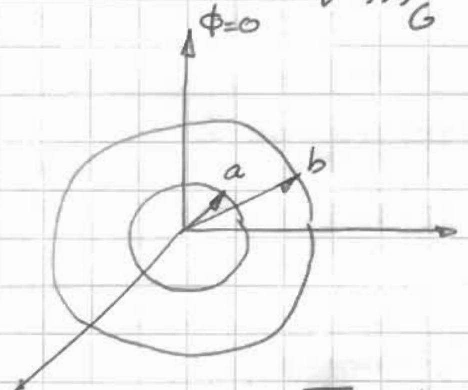
all integrals have constant limits

$$V = \int_{\pi/4}^{\tan^{-1}(2)} \int_0^{\pi/2} \int_0^1 r^2 \sin\phi \, dr\, d\phi\, d\theta$$

$$= \int_{\pi/4}^{\tan^{-1}(2)} \int_0^{\pi/2} \frac{r^3}{3} \sin\phi \Big|_0^1 d\phi d\theta = \frac{1}{3}\int_{\pi/4}^{\tan^{-1}(2)} (-\cos\phi)\Big|_0^{\pi/2} d\theta$$

$$V = \frac{1}{3}\left(\tan^{-1}(2) - \frac{\pi}{4}\right)(1)$$

9) find $\overline{T} = \frac{1}{V}\iiint_G T\,dV$ $\qquad G \quad \left.\begin{matrix} a \le r \le b \\ 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{matrix}\right\}$ spherical shell



$V_{sphere} = \frac{4}{3}\pi R^3 \leftarrow$ Schaums

$V = \frac{4}{3}\pi(b^3 - a^3)$

symmetry about origin integrate in spherical coords.

$$\overline{T} = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi}\int_a^b \underbrace{\left(T_0 + \frac{S(r^2-a^2)}{6k} + \frac{Sb^3}{3k}\left(\frac{1}{r} - \frac{1}{a}\right)\right)}_{\text{since no } \phi,\, \theta \text{ terms appear evaluate middle, outer integrals}} r^2 \sin\phi \, dr\, d\phi\, d\theta$$

$$\overline{T} = \frac{2\pi}{V}\int_a^b \left(T_0 + \frac{S(r^2-a^2)}{6k} + \frac{Sb^3}{3k}\left(\frac{1}{r} - \frac{1}{a}\right)\right)(-\cos\phi)\Big|_0^{\pi} r^2\, dr$$

$$= \frac{4\pi}{\frac{4}{3}\pi(b^3-a^3)}\int_a^b T(r)\, r^2\, dr = \frac{3}{(b^3-a^3)}\int_a^b T(r)\, r^2\, dr$$

b)

Archimedes principle: mass of body = mass of water displaced. (17)

$$m_b = V_b \cdot \rho_b$$

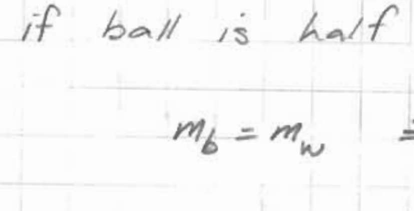
$$m_w = V_w \cdot \rho_w$$

if ball is half submerged, $V_w = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right) = \frac{1}{2} V_b$

$$m_b = m_w \Rightarrow V_b \cdot \rho_b = V_w \cdot \rho_w$$

$$V_b \cdot \rho_b = \frac{1}{2} V_b \cdot \rho_w \quad \therefore \rho_b = \frac{\rho_w}{2}$$

force required to submerge ball

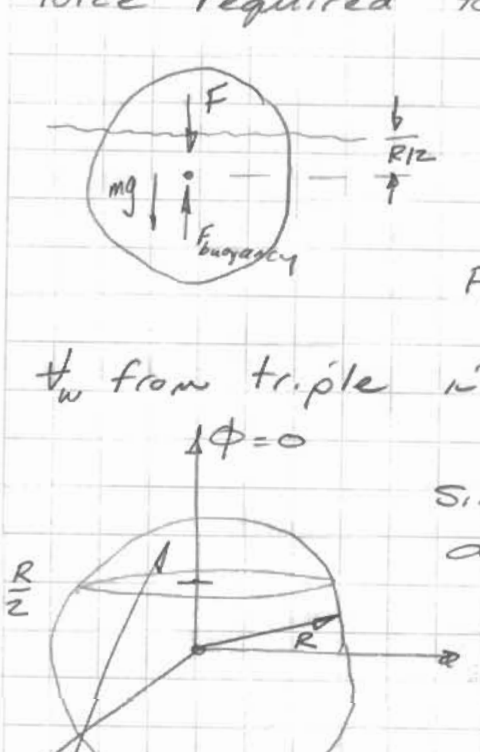


Force balance $F_{buoyancy} - mg = F_{required}$

$$mg = V_b \cdot \rho_b \cdot g$$

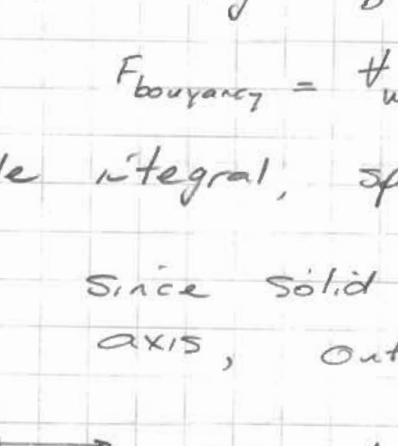
$$F_{bouyancy} = V_w \cdot \rho_w \cdot g$$

V_w from triple integral, spherical coordinates.



Since solid is axisymmetric about vertical axis, Outer integral $0 \le \theta \le 2\pi$

reduce to 2D integral in r, ϕ



$V_w = \frac{4}{3}\pi R^3 - V_{cap}$

(easier to evaluate this volume).

convert equations to spherical coordinates

$$z = \frac{R}{2} = r\cos\phi$$

$$r = \frac{R}{2} \cdot \frac{1}{\cos\phi}$$

inner integral $\frac{R}{2} \cdot \frac{1}{\cos\phi} \le r \le R$

find intersection $\frac{R}{2} \cdot \frac{1}{\cos\phi} = R \quad \cos\phi = \frac{1}{2} \quad \phi = \pi/3$

middle integral $0 \le \phi \le \pi/3$

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\frac{R}{2}\frac{1}{\cos\phi}}^{R} r^2 \sin\phi \, dr\, d\phi\, d\theta$$

$u = \cos\phi$

$du = -\sin\phi\, d\phi$

$-\int \frac{du}{u^3} = +\frac{1}{2}\frac{1}{u^2}$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{r^3}{3}\sin\phi \Big|_{\frac{R}{2}\frac{1}{\cos\phi}}^{R} d\phi\, d\theta = \frac{R^3}{3}\int_0^{2\pi}\int_0^{\pi/3} \sin\phi - \frac{1}{8}\frac{\sin\phi}{\cos^3\phi}\, d\phi\, d\theta$$

$$= \frac{R^3}{3}\int_0^{2\pi} -\cos\phi - \frac{1}{16}\cdot\frac{1}{\cos^2\phi}\Big|_0^{\pi/3} d\theta$$

$$= \frac{R^3}{3}(2\pi)\left(-\frac{1}{2} - \frac{1}{16}\cdot\frac{1}{(1/2)^2} + 1 + \frac{1}{16}\cdot 1\right) = \frac{5\pi}{24}R^3$$

$$\therefore\ F = V_w \cdot \rho_w \cdot g - V_b \cdot \rho_b \cdot g$$

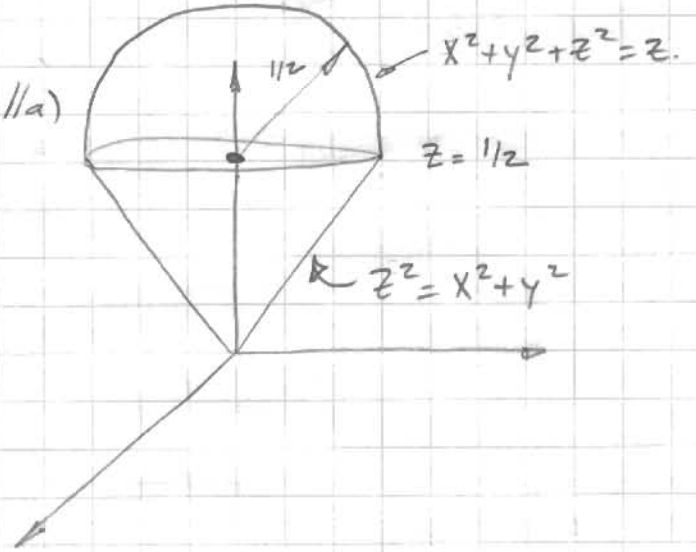
$$V_w = \frac{4}{3}\pi R^3 - \frac{5\pi}{24}R^3 = \frac{9\pi}{8}R^3$$

$$\rho_b = \frac{\rho_w}{2}$$

$$F = \frac{9\pi}{8}R^3 \cdot \rho_w \cdot g - \frac{4}{3}\pi R^3 \cdot \frac{\rho_w}{2}\cdot g$$

$$= \left(\frac{9\pi}{8} - \frac{4\pi}{6}\right) R^3 \rho_w g = \frac{11}{24}\pi R^3 \rho_w g$$

11a)



$z^2 = x^2+y^2 \Leftarrow$ cone

$$x^2+y^2+z^2 = z$$

$$x^2+y^2+z^2-z = 0$$

$$x^2+y^2+z^2-z+1/4 = 1/4$$

$$\underbrace{x^2+y^2+(z-1/2) = 1/4}_{}$$

sphere, center at $(0,0,1/2)$

radius $= 1/2$

intersection between surfaces

$$\underbrace{x^2+y^2 = z^2 = z - z^2}$$

$$2z^2 = z$$

$$2z = 1 \qquad z = 1/2$$

$\therefore$ intersection = circular disk, $r = 1/2$ at $z = 1/2$

b) convert surface equations to spherical coordinates

$x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$

$$x^2+y^2+z^2 = z$$

$$r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi = r\cos\phi$$

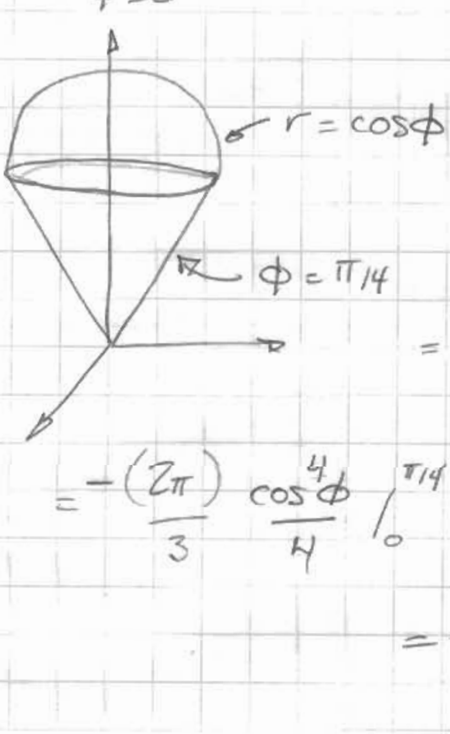
$$r^2 = r\cos\phi$$

$$r = \cos\phi$$

$$z^2 = x^2+y^2$$

$$r^2\cos^2\phi = r^2\sin^2\phi(\cos^2\theta+\sin^2\theta)$$

$$\frac{\sin\phi}{\cos\phi} = \tan\phi = 1 \qquad \phi = \pi/4$$



$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} r^2\sin\phi \, dr\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{r^3}{3}\sin\phi \Big|_0^{\cos\phi} d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \underbrace{\frac{\cos^3\phi\sin\phi\, d\phi\, d\theta}{3}}$$

$$= -\frac{(2\pi)}{3}\frac{\cos^4\phi}{4}\Big|_0^{\pi/4} = -\frac{2\pi}{3}\cdot\frac{1}{4}\cdot\left[\left(\frac{1}{\sqrt{2}}\right)^4 - 1\right]$$

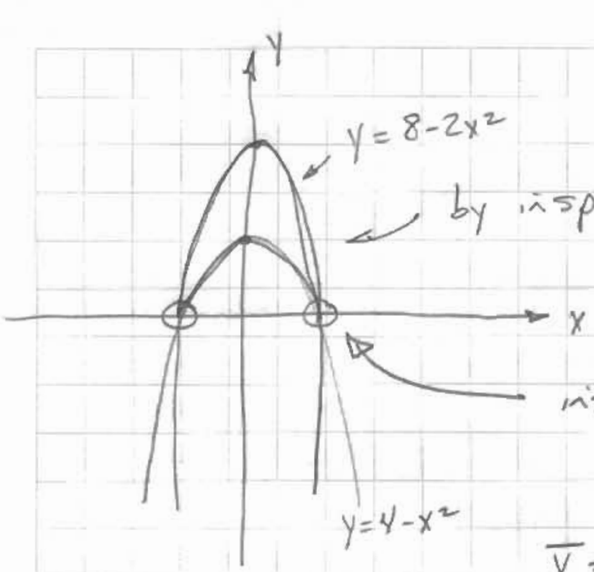
$u = \cos\phi$
$du = -\sin\phi\, d\phi$
$-\int u^3 du = \frac{u^4}{2}$

$$= -\frac{\pi}{6}\left(\frac{1}{4}-1\right) = \frac{\pi}{8}$$

12a) $\bar{x} = \frac{1}{M}\iint_R x\rho^* dA$

$R: \; y = 8-2x^2$
$\quad\; y = 4-x^2$

$\bar{y} = \frac{1}{M}\iint_R y\rho^* dA$



by inspection: $\bar{x} = 0$ because R is symmetric about y axis.

intersection $y = 8-2x^2 = 4-x^2$

$4 = x^2 \quad x = \pm 2$

$$\bar{y} = \frac{1}{M}\int_{-2}^{2}\int_{4-x^2}^{8-2x^2} y\,\rho^*\,dy\,dx$$

$$= \frac{\rho^*}{M}\int_{-2}^{2} \frac{y^2}{2}\Big|_{4-x^2}^{8-2x^2} dx = \frac{\rho^*}{2M}\int_{-2}^{2} 64-32x^2+4x^4-16+8x^2-x^4\,dx$$

$$= \frac{\rho^*}{2M}\int_{-2}^{2} 48-24x^2+3x^4\,dx = \frac{\rho^*}{2M}\left[48x-8x^3+\frac{3}{5}x^5\right]\Big|_{-2}^{2}$$

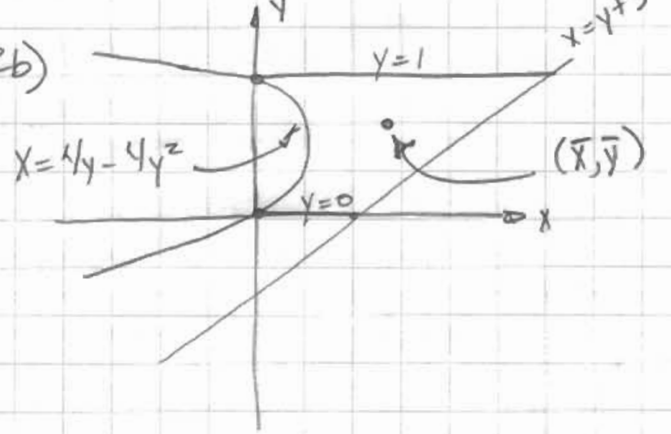
$$= \frac{\rho^*}{2M}\left[\frac{512}{5}\right] = \frac{\rho^*}{M}\cdot\frac{256}{5}$$

$$M = \int_{-2}^{2}\int_{4-x^2}^{8-2x^2} \rho^*\,dy\,dx = \rho^*\int_{-2}^{2} 8-2x^2-4+x^2\,dx$$

$$= \rho^*\int_{-2}^{2} 4-x^2\,dx = \rho^*\left[4x-\frac{x^3}{3}\right]\Big|_{-2}^{2} = \frac{32}{3}\rho^*$$

$$\bar{y} = \frac{\rho^*}{\left(\frac{32}{3}\rho^*\right)}\cdot\frac{256}{5} = \frac{3}{32}\cdot\frac{256}{5} = \frac{24}{5} = 4.8$$

12b)



$X = 4y - 4y^2$ ← parabola

$X = y+3$

$y = 0, \quad y = 1$

y	X
1	0
2	-8
0	0
-1	-8

$$\bar{X} = \frac{1}{M}\iint_R x\rho^* \, dA \qquad \bar{X} = \frac{1}{M}\int_0^1\int_{4y-4y^2}^{y+3} x\rho^* \, dx\,dy$$

$$\bar{Y} = \frac{1}{M}\iint_R y\rho^* \, dA$$

$$M = \iint_R \rho^* \, dA \qquad = \frac{\rho^*}{2M}\int_0^1 x^2 \Big|_{4y-4y^2}^{y+3} dy$$

$$= \frac{\rho^*}{2M}\int_0^1 y^2+6y+9 - 16y^2+32y^3-16y^4 \, dy$$

$$= \frac{\rho^*}{2M}\int_0^1 -16y^4+32y^3-15y^2+6y+9 \, dy = \frac{\rho^*}{2M}\left[-\frac{16y^5}{5}+8y^4-5y^3+3y^2+9y\right]\Big|_0^1$$

$$= \frac{\rho^*}{2M}\,\frac{59}{5} = \frac{\rho^*}{M}\,\frac{59}{10}$$

$$\bar{Y} = \frac{1}{M}\int_0^1\int_{4y-4y^2}^{y+3} y\rho^* \, dx\,dy = \frac{\rho^*}{M}\int_0^1 y^2+3y-4y^2+4y^3 \, dy$$

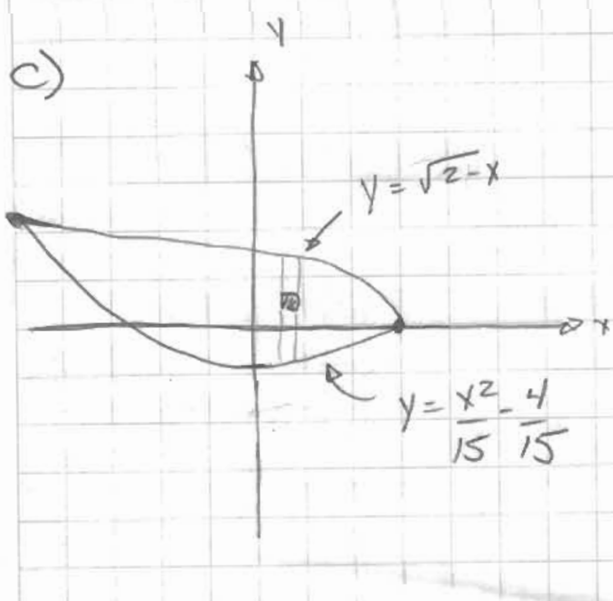
$$= \frac{\rho^*}{M}\left[\frac{3y^2}{2} - \frac{3y^3}{3} + y^4\right]\Big|_0^1 = \frac{\rho^*}{M}\cdot\frac{3}{2}$$

$$M = \int_0^1\int_{4y-4y^2}^{y+3}\rho^* \, dx\,dy = \rho^*\int_0^1 y+3-4y+4y^2 \, dy = \rho^*\left[3y - \frac{3y^2}{2}+\frac{4y^3}{3}\right]\Big|_0^1$$

$$= \rho^*\cdot\frac{17}{6}$$

$$\bar{X} = \frac{6}{17}\cdot\frac{59}{10} = \frac{177}{85} \qquad \bar{Y} = \frac{6}{17}\cdot\frac{3}{2} = \frac{18}{34}$$

c)



parabola

$y = \sqrt{2-x} \qquad x = 2-y^2, \ y \geq 0$

$y = \frac{x^2}{15} - \frac{4}{15}$

find intersection points

$$\sqrt{2-x} = \frac{x^2}{15} - \frac{4}{15}$$

$$2-x = \frac{1}{225}(x^4-8x^2+16)$$

$x^4 - 8x^2 + 225x - 434 = 0$ $\quad$ $\underbrace{x = 2,\ -7}$

from Maple solution
(or Mathcad)
ignore complex roots.

$$\overline{X} = \frac{\rho^*}{M} \int_{-7}^{2} \int_{\frac{x^2}{15}-\frac{4}{15}}^{\sqrt{2-x}} x \, dy dx$$

$u = 2 - x \qquad du = -dx$

$-\int (2-u)\sqrt{u}\, du$

$= -\int 2\sqrt{u}\, du + \int u^{3/2} du$

$= -\frac{4}{3} u^{3/2} + \frac{5}{2} u^{5/2}$

$$= \frac{\rho^*}{M} \int_{-7}^{2} x\sqrt{2-x} - \frac{x^3}{15} + \frac{4x}{15}\, dx$$

$$= \frac{\rho^*}{M} \left(-\frac{4}{3}(2-x)^{3/2} + \frac{5}{2}(2-x)^{5/2} - \frac{x^4}{60} + \frac{2x^2}{15} \right) \Big|_{-7}^{2}$$

$$= \frac{\rho^*}{M} \cdot \left(-\frac{549}{20} \right)$$

$$\overline{Y} = \frac{\rho^*}{M} \int_{-7}^{2} \int_{\frac{x^2}{15}-\frac{4}{15}}^{\sqrt{2-x}} y \, dy dx = \frac{\rho^*}{M} \int_{-7}^{2} \frac{1}{2} \left(2 - x - \frac{x^4}{225} + \frac{8x^2}{225} - \frac{16}{225} \right) dx$$

$$= \frac{\rho^*}{M} \left[x - \frac{x^2}{4} - \frac{x^5}{2250} + \frac{8x^3}{1350} - \frac{8x}{225} \right] \Big|_{-7}^{2} = \frac{\rho^*}{M} \cdot \frac{65367}{4500}$$

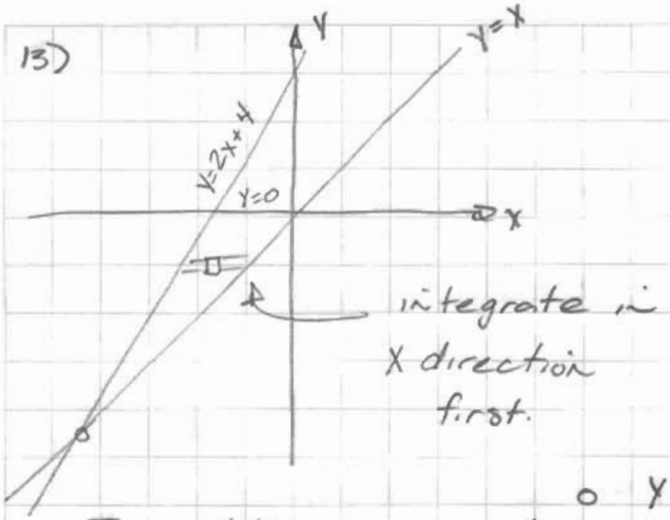
$$M = \rho^* \int_{-7}^{2} \int_{\frac{x^2}{15}-\frac{4}{15}}^{\sqrt{2-x}} dy dx = \rho^* \int_{-7}^{2} \sqrt{2-x} - \frac{x^2}{15} + \frac{4}{15}\, dx$$

$u = 2 - x$
$du = -dx$
$-\int \sqrt{u}\, du = -\frac{2}{3} u^{3/2}$

$$= \rho^* \left[-\frac{2}{3}(2-x)^{3/2} - \frac{x^3}{45} + \frac{4x}{15} \right] \Big|_{-7}^{2} = \rho^* \cdot \frac{63}{5}$$

$$\overline{X} = \frac{5}{63} \cdot \left(-\frac{549}{20} \right) = \frac{-61}{28} \qquad \overline{Y} = \frac{5}{63} \cdot \left(\frac{65367}{4500} \right) = \frac{807}{700}$$

13)



inner $\frac{y-4}{2} \le x \le y$

outer intersection

$y = x = 2x+4$

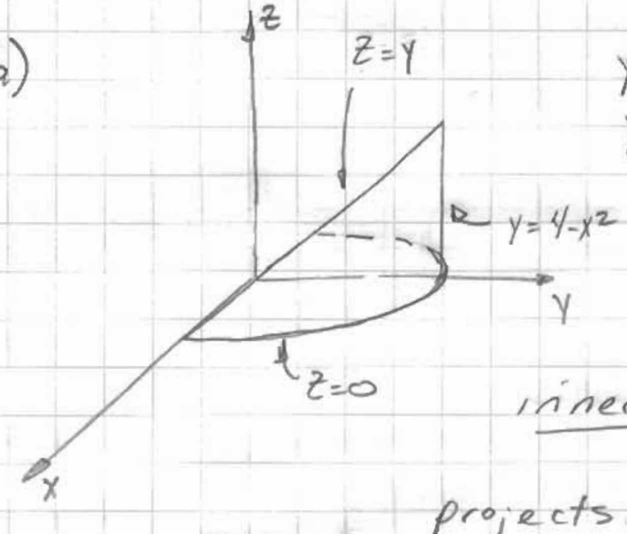
$x = -4$, $y = -4$

outer $-4 \le y \le 0$

$$I_x = \rho^* \iint_R y^2\, dA = \rho^* \int_{-4}^{0} \int_{\frac{y-4}{2}}^{y} y^2\, dx\, dy = \rho^* \int_{-4}^{0} y^3 - \frac{y^3}{2} + 2y^2\, dy$$

$$= \rho^* \left(\frac{y^4}{8} + \frac{2y^3}{3} \right) \Big|_{-4}^{0} = \rho^* \left(\frac{2(4)^3}{3} - \frac{(4)^4}{8} \right) = \frac{32}{3} \cdot \rho^*$$

14a)



$y = 4 - x^2$ ← paraboloid

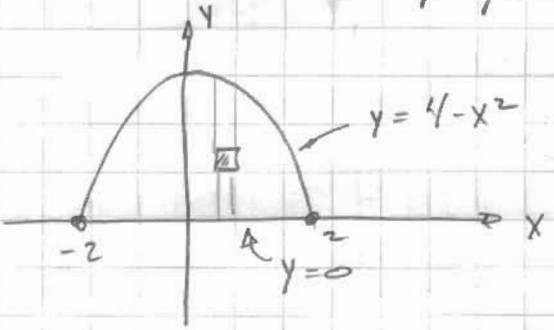
$y = z$

$z = 0$

first integration in z direction

inner integral $0 \le z \le y$

projects onto xy plane



middle $0 \le y \le 4 - x^2$

outer $-2 \le x \le 2$

by inspection $\bar{X} = 0$

$$\bar{y} = \frac{1}{M} \iiint_G y \rho\, dV = \frac{\rho}{M} \int_{-2}^{2} \int_{0}^{4-x^2} \int_{0}^{y} y\, dz\, dy\, dx$$

$$= \frac{\rho}{M} \int_{-2}^{2} \int_{0}^{4-x^2} y^2\, dy\, dx = \frac{\rho}{M} \int_{-2}^{2} \frac{(4-x^2)^3}{3}\, dx$$

$$= \frac{\rho}{3M}\int_{-2}^{2} (4-x^2)(16-8x^2+x^4)\,dx = \frac{\rho}{3M}\int_{-2}^{2}(64-32x^2+4x^4-16x^2+8x^4-x^6)\,dx$$

$$= \frac{\rho}{3M}\left(64x - \frac{48x^3}{3} + \frac{12x^5}{5} - \frac{x^7}{7}\right)\Big|_{-2}^{2} = \frac{4096}{105}\rho$$

$$\bar{z} = \frac{1}{M}\iiint_G \rho z\, dV = \frac{\rho}{M}\int_{-2}^{2}\int_{0}^{4-x^2}\int_{0}^{y} z\,dz\,dy\,dx$$

$$= \frac{\rho}{2M}\int_{-2}^{2}\int_{0}^{4-x^2} y^2\,dy\,dx = \frac{\rho}{6M}\int_{-2}^{2}(4-x^2)^3\,dx = \frac{\bar{y}}{2} \quad \therefore \bar{z} = \frac{2048\rho}{105}$$

$$M = \int_{-2}^{2}\int_{0}^{4-x^2}\int_{0}^{y} \rho\,dz\,dy\,dx = \rho\int_{-2}^{2}\int_{0}^{4-x^2} y\,dy\,dx$$

$$= \frac{\rho}{2}\int_{-2}^{2}(4-x^2)^2\,dx = \frac{\rho}{2}\int_{-2}^{2} 16-8x^2+x^4\,dx$$

$$= \frac{\rho}{2}\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\Big|_{-2}^{2} = \frac{256}{15}\rho$$

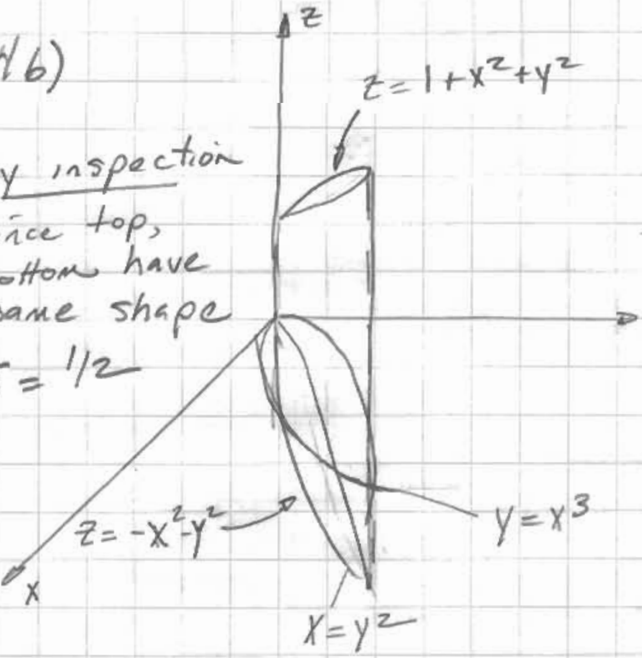
$$\therefore \bar{x} = 0, \quad \bar{y} = \frac{15}{256}\cdot\frac{4096}{105} = \frac{16}{7}, \quad \bar{z} = \frac{15}{256}\cdot\frac{2048}{105} = \frac{8}{7}$$

46)

by inspection

since top, bottom have same shape

$\bar{z} = 1/2$



$\left.\begin{aligned} y &= x^3 \\ x &= y^2 \end{aligned}\right\}$ paraboloids

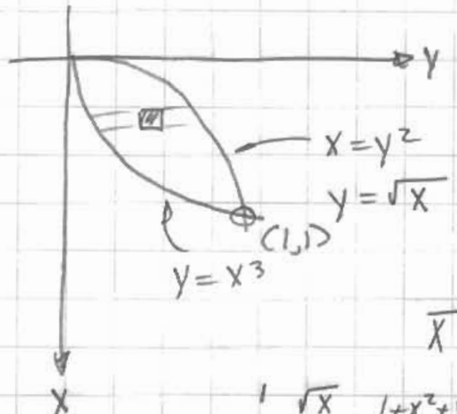
$\left.\begin{aligned} z &= 1+x^2+y^2 \\ z &= -(x^2+y^2) \end{aligned}\right\}$ paraboloids

Since cross sectional shape constant in z direction, integrate z first

inner $\quad -x^2-y^2 \le z \le 1+x^2+y^2$

examine xy plane. middle integral

$$x^3 \le y \le \sqrt{x}$$



outer integral

$$0 \le x \le 1$$

$$\bar{X} = \frac{1}{M}\iiint_G \rho x\, dV$$

$$= \frac{\rho}{M}\int_0^1\int_{x^3}^{\sqrt{x}}\int_{-x^2-y^2}^{1+x^2+y^2} x\, dz\, dy\, dx = \frac{\rho}{M}\int_0^1\int_{x^3}^{\sqrt{x}} x+x^3+xy^2+x^3+xy^2\, dy\, dx$$

$$= \frac{\rho}{M}\int_0^1 xy+2x^3y+\frac{2xy^3}{3}\Big|_{x^3}^{\sqrt{x}}$$

$$= \frac{\rho}{M}\int_0^1 x^{3/2}+2x^{7/2}+\frac{2}{3}x^{5/2}-x^4-2x^6-\frac{2}{3}x^{10}\, dx$$

$$= \frac{\rho}{M}\left(\frac{2}{5}x^{5/2}+\frac{4}{9}x^{9/2}+\frac{4}{21}x^{7/2}-\frac{x^5}{5}-\frac{2x^7}{7}-\frac{2}{33}x^{11}\right)\Big|_0^1 = \frac{\rho}{M}\cdot\frac{1693}{3465}$$

$$\bar{Y} = \frac{\rho}{M}\int_0^1\int_{x^3}^{\sqrt{x}}\int_{-x^2-y^2}^{1+x^2+y^2} y\, dz\, dy\, dx = \frac{\rho}{M}\int_0^1\int_{x^3}^{\sqrt{x}} y+x^2y+y^3+x^2y+y^3\, dy\, dx$$

$$= \frac{\rho}{M}\int_0^1 \frac{y^2}{2}+x^2y^2+\frac{y^4}{2}\Big|_{x^3}^{\sqrt{x}} dx = \frac{\rho}{M}\int_0^1 \frac{x}{2}+x^3+\frac{x^2}{2}-\frac{x^6}{2}-x^8-\frac{x^{12}}{2}\, dx$$

$$= \frac{\rho}{M}\left(\frac{x^2}{4}+\frac{x^4}{4}+\frac{x^3}{6}-\frac{x^7}{14}-\frac{x^9}{9}-\frac{x^{13}}{26}\right)\Big|_0^1 = \frac{\rho}{M}\cdot\frac{365}{819}$$

$$M = \rho\int_0^1\int_{x^3}^{\sqrt{x}}\int_{-x^2-y^2}^{1+x^2+y^2} dz\, dy\, dx = \rho\int_0^1\int_{x^3}^{\sqrt{x}} 1+2x^2+2y^2\, dy\, dx$$

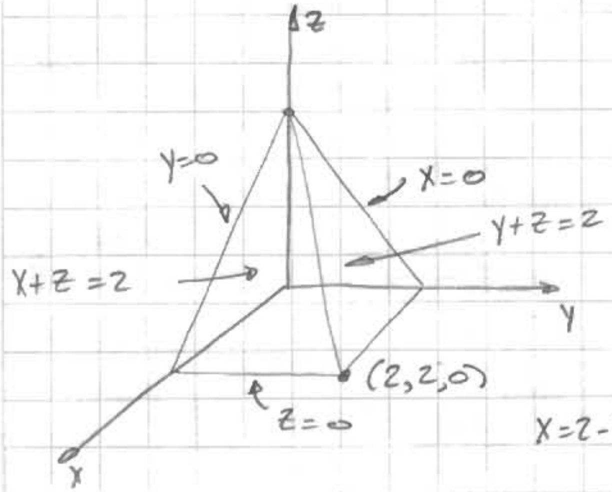
$$= \rho\int_0^1 y+2x^2y+\frac{2y^3}{3}\Big|_{x^3}^{\sqrt{x}} dx = \rho\int_0^1 \sqrt{x}+2x^{5/2}+\frac{2}{3}x^{3/2}-x^3-2x^5-\frac{2}{3}x^9\, dx$$

$$= \rho\left(\frac{2}{3}x^{3/2}+\frac{4}{7}x^{7/2}+\frac{4}{15}x^{5/2}-\frac{x^4}{4}-\frac{2x^6}{6}-\frac{2x^{10}}{30}\right)\Big|_0^1 = \frac{359}{420}\rho$$

$$\overline{X} = \frac{420}{359} \cdot \frac{1693}{3465} = \frac{6772}{11847} \qquad \overline{z} = \frac{1}{2} \text{ (by inspection)}$$

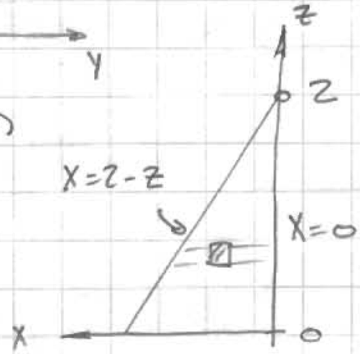
$$\overline{Y} = \frac{420}{359} \cdot \frac{365}{819} = \frac{7300}{14001}$$

15) $I_z = \iiint_G (x^2+y^2)\,\rho\, dV$ $\qquad G: \; y+z=2 \quad x=0$
$\qquad\qquad x+z=2 \quad y=0$
$\qquad\qquad\qquad z=0$



integrate in y direction first

inner $\quad 0 \le y \le 2-z$ $\quad$ project on xz plane



middle $\quad 0 \le x \le 2-z$

outer $\quad 0 \le z \le 2$

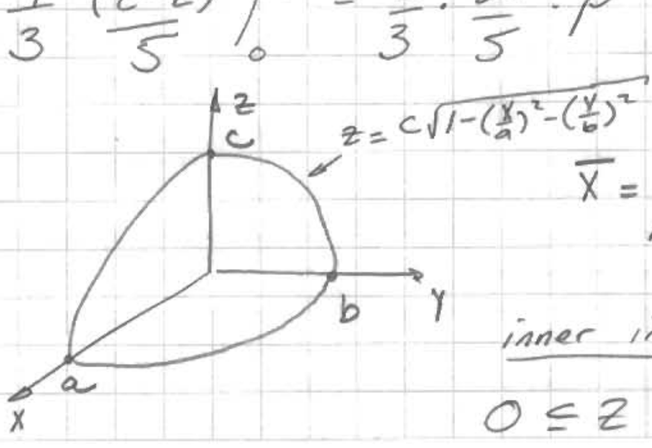
$$I_z = \rho \int_0^2 \int_0^{2-z} \int_0^{2-z} (x^2+y^2)\,dy\,dx\,dz = \rho \int_0^2 \int_0^{2-z} x^2 y + \frac{y^3}{3} \Big|_0^{2-z} dx\,dz$$

$$= \rho \int_0^2 \int_0^{2-z} x^2(2-z) + \frac{(2-z)^3}{3}\,dx\,dz = \rho \int_0^2 \frac{x^3}{3}(2-z) + \frac{(2-z)^3}{3} x \Big|_0^{2-z} dz$$

$$= \rho \int_0^2 \frac{2}{3}(2-z)^4 dz \qquad \begin{matrix} u = 2-z \\ du = -dz \end{matrix} \quad -\int u^4 du = -\frac{u^5}{5}$$

$$= -\frac{2\rho}{3} \frac{(2-z)^5}{5} \Big|_0^2 = \frac{2}{3} \cdot \frac{32}{5} \cdot \rho = \frac{64\rho}{15}$$

16)

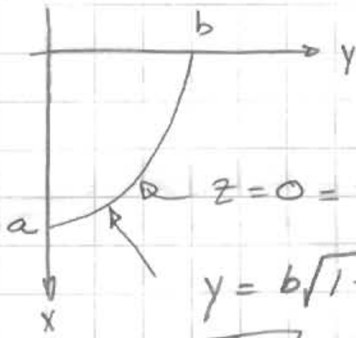


$$\overline{X} = \frac{1}{M} \iiint_G \rho x\, dV$$

inner integral

$$0 \le z \le c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}$$

↳ project onto xy plane.



$z = 0 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$

$y = b\sqrt{1-\left(\frac{x}{a}\right)^2}$

middle integral

$0 \le y \le b\sqrt{1-\left(\frac{x}{a}\right)^2}$

outer integral

$0 \le x \le a$

$$\bar{x} = \frac{\rho}{M} \int_0^a \int_0^{b\sqrt{1-(\frac{x}{a})^2}} \int_0^{c\sqrt{1-(\frac{x}{a})^2-(\frac{y}{b})^2}} x \, dz\,dy\,dx = \frac{\rho}{M}\int_0^a \int_0^{b\sqrt{1-(\frac{x}{a})^2}} x c\sqrt{1-\left(\tfrac{x}{a}\right)^2-\left(\tfrac{y}{b}\right)^2}\, dy\,dx$$

↳ solutions gets messy at this point

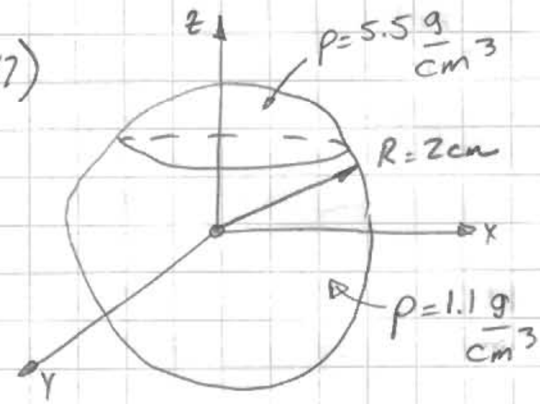
$$\bar{x} = \rho\pi a^2 bc / (16M)$$

$$M = \rho \int_0^a \int_0^{b\sqrt{1-(\frac{x}{a})^2}} \int_0^{c\sqrt{1-(\frac{x}{a})^2-(\frac{y}{b})^2}} dz\,dy\,dz = \frac{\rho\pi abc}{6}$$

$$\bar{x} = \frac{6}{\rho\pi abc} \cdot \frac{\rho\pi a^2 bc}{16} = \frac{3}{8}a$$

$\bar{y}, \bar{z}$ similar $\quad \bar{y} = \frac{3}{8}b, \quad \bar{z} = \frac{3}{8}c.$

17)

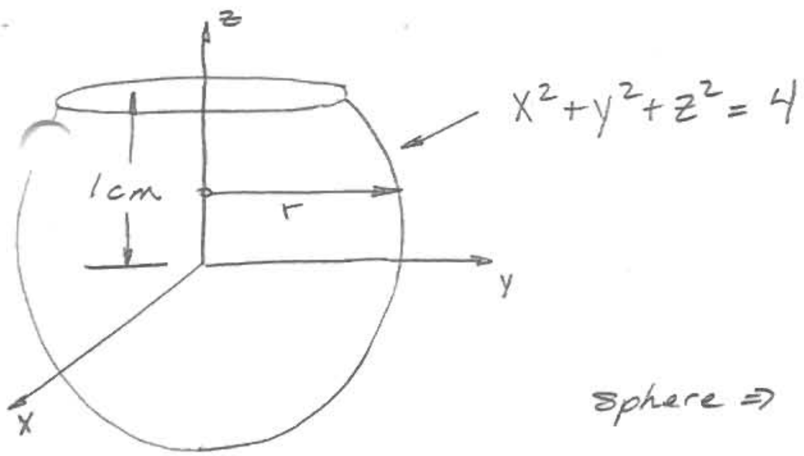


by inspection (symmetry) $\bar{x} = 0$, $\bar{y} = 0$

$$\bar{z} = \frac{1}{M}\iiint_G \rho z \, dV$$

$$= \frac{1}{M}\iiint_{V_1} z\rho_1 \, dV_1 + \frac{1}{M}\iiint_{V_2} z\rho_2\, dV_2$$

core, $\rho = 1100 \frac{kg}{m^3}$ $\quad$ cap, $\rho = 5500$



symmetric about z axis
$\therefore$ integration in polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

sphere $\Rightarrow$ $r^2+z^2=4$

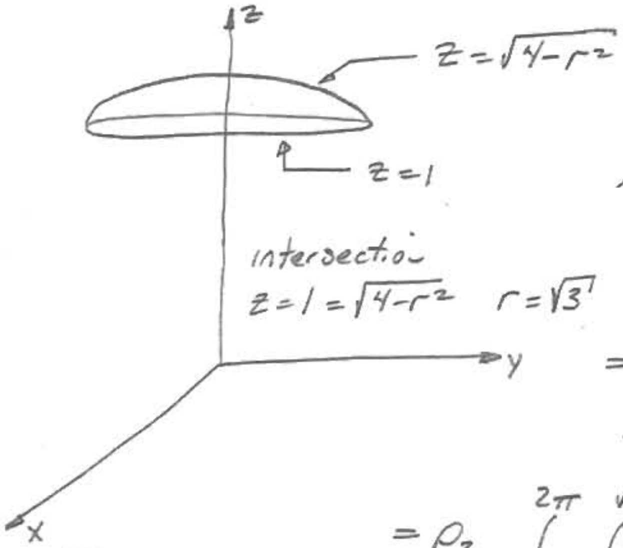
$$r=\sqrt{4-z^2}$$

$\therefore$ volume integral $\displaystyle\iiint_{V_1} z\rho_1\, dV_1 = \rho\int_0^{2\pi}\int_{-2}^{1}\int_0^{\sqrt{4-z^2}} z\, r\, dr\, dz\, d\theta$

$$= \rho_1\int_0^{2\pi}\int_{-2}^{1} z\frac{r^2}{2}\Big|_0^{\sqrt{4-z^2}} dz\, d\theta = \frac{\rho_2}{2}\int_0^{2\pi}\int_{-2}^{1} z(4-z^2)\, dz\, d\theta$$

$$= \frac{\rho_1}{2}\int_0^{2\pi} \frac{4z^2}{2} - \frac{z^4}{4}\Big|_{-2}^{1} d\theta$$

$$= 2\pi\left(\frac{\rho_1}{2}\right)\left[2-\frac{1}{4} - \frac{4(4)}{2}+\frac{16}{4}\right] = -\frac{9}{4}\pi\rho_1$$



intersection
$z=1=\sqrt{4-r^2}$ $\quad r=\sqrt{3}$

$$\iiint_{V_1} \rho_1 z\, V_1 = \rho_1\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} z\, r\, dz\, dr\, d\theta$$

$$= \frac{\rho_2}{2}\int_0^{2\pi}\int_0^{\sqrt{3}} r\, z^2\Big|_1^{\sqrt{4-r^2}} dr\, d\theta$$

$$= \frac{\rho_2}{2}\int_0^{2\pi}\int_0^{\sqrt{3}} 4r - r^3 - r\, dr\, d\theta$$

$$= \frac{\rho_2}{2}\int_0^{2\pi} \frac{3r^2}{2} - \frac{r^4}{4}\Big|_0^{\sqrt{3}} d\theta$$

$$= \frac{\rho_2}{2} \cdot 2\pi \cdot \left(\frac{9}{2} - \frac{9}{4} \right) = \rho_2 \pi \frac{9}{4}$$

$$\therefore \bar{z} = \frac{-\frac{9}{4}\pi\rho_1 + \frac{9}{4}\pi\rho_2}{M}$$

$$M = \iiint_{V_1} \rho_1 \, dV_1 + \iiint_{V_2} \rho_2 \, dV_2$$

$$= \rho_1 \int_0^{2\pi} \int_{-2}^{1} \int_0^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta + \rho_2 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

$$= \frac{\rho_1}{2}(2\pi) \int_{-2}^{1} (4-z^2) \, dz + \rho_2 (2\pi) \int_0^{\sqrt{3}} r\sqrt{4-r^2} - r \, dr$$

Schaums $\int x\sqrt{a^2-x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{3}$

$$= \rho_1 \pi \left[4z - \frac{z^3}{3} \right] \Big|_{-2}^{1} + 2\pi\rho_2 \left[-\frac{(4-r^2)^{3/2}}{3} - \frac{r^2}{2} \right] \Big|_0^{\sqrt{3}}$$

$$= \rho_1 \pi \left[4 - \frac{1}{3} + 8 - \frac{8}{3} \right] + 2\pi\rho_2 \left[-\frac{1}{3} - \frac{3}{2} + \frac{4^{3/2}}{3} \right]$$

$$= 9\pi\rho_1 + \frac{5}{3}\pi\rho_2$$

$$\therefore \bar{z} = \frac{-\frac{9}{4}\pi\rho_1 + \frac{9}{4}\pi\rho_2}{9\pi\rho_1 + \frac{5}{3}\pi\rho_2} = \frac{-\frac{9}{4} + \frac{9}{4}(5)}{9 + \frac{5}{3}(5)}$$

$$= \frac{9}{52 / 3} = 9 \cdot \frac{3}{52} = \frac{27}{52}$$

$$\bar{z} \cong \cancel{0.730} \text{ cm}$$

$$0.519 \text{ cm}$$