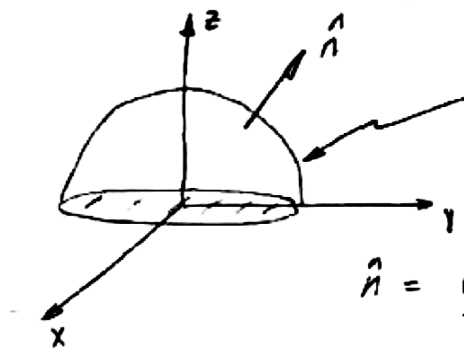
Tutorial #10 Solutions

1) $\vec{F} = 0\hat{i} + 0\hat{j} + (3z+1)\hat{k}$ $\quad$ <u>hemisphere</u>



$x^2+y^2+z^2=a^2 \Rightarrow z = \sqrt{a^2-x^2-y^2}$

$$\vec{n} = \nabla(x^2+y^2+z^2-a^2) = (2x, 2y, 2z)$$

$$\hat{n} = \frac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \frac{(x,y,z)}{a}$$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S (0,0,3z+1)\cdot\frac{(x,y,z)}{a}\,dS$$

$$= \iint_S \frac{3z^2+z}{a}\,dS$$

project to xy plane

$$= \frac{1}{a}\iint_{R_{xy}} \left(3(a^2-x^2-y^2) + \sqrt{a^2-x^2-y^2}\right)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

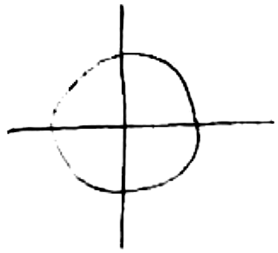
$$\frac{\partial\left(\sqrt{a^2-x^2-y^2}\right)}{\partial x} = \frac{1}{2}\frac{(-2x)}{\sqrt{a^2-x^2-y^2}} = \frac{-x}{\sqrt{a^2-x^2-y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$$= \frac{1}{a}\iint_{R_{xy}} \left(3(a^2-x^2-y^2) + \sqrt{a^2-x^2-y^2}\right)\sqrt{1+\frac{x^2+y^2}{a^2-x^2-y^2}}\,dA$$

$$= \frac{1}{a}\iint_{R_{xy}} \left(3(a^2-x^2-y^2)^{1/2} + \sqrt{a^2-x^2-y^2}\right)\sqrt{\frac{a^2-x^2-y^2+x^2+y^2}{a^2-x^2-y^2}}\,dA$$

$$= \iint_{R_{xy}} 3\sqrt{a^2-x^2-y^2}+1\,dA$$

on xy plane. – circle – easier to evaluate in polar coordinates



$x = r\cos\theta$

$y = r\sin\theta$

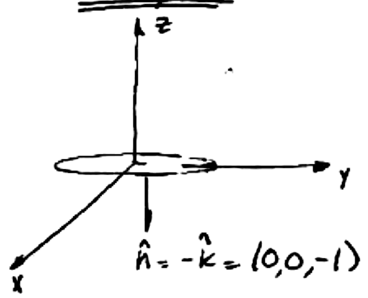
$\therefore x^2+y^2 = r^2$

$$= 3\int_0^{2\pi}\int_0^{a} \sqrt{a^2-r^2}\, r\,dr\,d\theta + \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$$

$$= -(2\pi)(a^2-r^2)^{3/2}\Big|_0^a + (2\pi)\frac{r^2}{2}\Big|_0^a$$

$$= 2\pi a^3 + \pi a^2$$

on xy plane



$z = 0 \;,\; \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S -3z-1\;dS$$

$$= -\iint_S dS$$

$$= -\iint_{R_{xy}} \sqrt{1+0+0}\;dA \quad = \quad -\int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$$

$$= -\pi a^2$$

$\therefore \oiint_S \vec{F}\cdot\hat{n} = 2\pi a^3 + \pi a^2 - \pi a^2 = 2\pi a^3$ outward (positive sign)

b) divergence theorem $\quad \oiint \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

$$\nabla\cdot\vec{F} = 0 + 0 + \frac{\partial}{\partial z}(3z+1) = 3$$

$3\iiint_V dV$ = 3 x volume of hemisphere

$$= 3\left(\frac{4}{3}\pi a^3\right)\left(\frac{1}{2}\right) = 2\pi a^3$$