ME 201 Tutorial #4 Solutions - Winter 2017
Chain Rule, Gradient Vector and Directional Derivative

1. The mass of a rocket lifting off from earth decreases due to fuel consumption at a rate of 40 *kg/s*. Using Newton's law of gravitation:

$$F = \frac{G\,M\,m}{r^2}$$

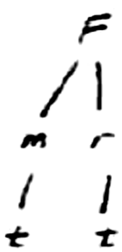
$G =$ universal gravitational constant $\left(km^3/kg\,s^2\right)$
$M =$ mass of the earth (kg)
$m =$ mass of rocket (including fuel) (kg)
$r =$ distance between rocket and center of the earth (km)

calculate how fast the magnitude F of the force of gravity is decreasing when the rocket is 6400 *km* from the center of the earth and is rising with a velocity of 100 *km/s*? (N/s)

given $F = \frac{GMm}{r^2}$ find $\frac{dF}{dt}$

G, M constants

m, r vary with time



$$\frac{dF}{dt} = \frac{\partial F}{\partial m}\cdot\frac{dm}{dt} + \frac{\partial F}{\partial r}\,\frac{dr}{dt}$$

$$\frac{\partial F}{\partial m} = \frac{GM}{r^2} \quad \left[\frac{km^3}{kg\,s^2}\cdot\frac{kg}{km^2}\right]$$

$$\frac{dm}{dt} = -40\,\frac{kg}{s} \leftarrow \text{mass } \underline{\text{decreasing}}$$

$$\frac{\partial F}{\partial r} = \frac{-2GMm}{r^3} \quad \left[\frac{km^3}{kg\,s^2}\cdot\frac{kg^2}{km^3}\right]$$

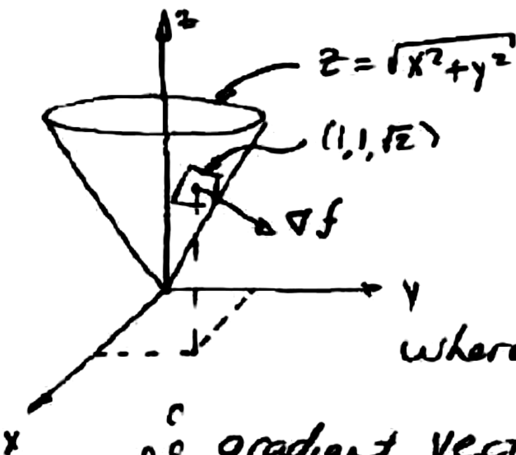
$$\frac{dr}{dt} = 100\,\frac{km}{s}$$

when $r = 6400\,km$ $\quad \frac{dF}{dt} = \frac{-GM}{(6400)^2}\cdot 40 - \frac{2GMm}{(6400)^3}\cdot 100 \left[\frac{km\ k}{s^3}\right.$

$$= \frac{-40GM}{(6400)^2}\left(1 + \frac{5m}{6400}\right)\frac{km\ kg}{s^2}\ \frac{1}{s} \times \frac{1000m}{km}$$

$$= -\frac{GM}{1024}\left(1 + \frac{5m}{6400}\right)\frac{N}{s}$$

2. Find the equation of the plane tangent to the surface of the cone $z = \sqrt{x^2 + y^2}$ at the point $(1, 1, \sqrt{2})$



equation of plane

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

where $(A, B, C) \perp$ to surface

∴ gradient vector ∇f where $f = z - \sqrt{x^2+y^2} = 0$

$$\nabla f = \left(\frac{\partial}{\partial x}\left(z - \sqrt{x^2+y^2}\right),\ \frac{-y}{\sqrt{x^2+y^2}},\ 1 \right)$$

$$\nabla f\Big|_{(1,1,\sqrt{2})} = \left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 1 \right) = (A, B, C)$$

if $(x_0, y_0, z_0) = (1, 1, \sqrt{2})$ then equation $\Rightarrow$

$$\left(\frac{-1}{\sqrt{2}}\right)(x-1) + \left(\frac{-1}{\sqrt{2}}\right)(y-1) + (1)(z - \sqrt{2}) = 0$$

$$\left[-\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} + z = \sqrt{2} - \frac{2}{\sqrt{2}} \right] \times (-\sqrt{2})$$

$$x + y - \sqrt{2}\,z = -2 + 2 = 0$$

3. Given the density function $\rho(x,y,z) = x^3y^2z^5 - 2xz + yz + 3x$ find the rate of change of the function at the following points in the direction given:

   a. In the direction of the most rapid density increase at $(1,2,-1)$

   b. At point $(2,2,1)$ in the direction away from the origin.

$\rho = x^3y^2z^5 - 2xz + yz + 3x$

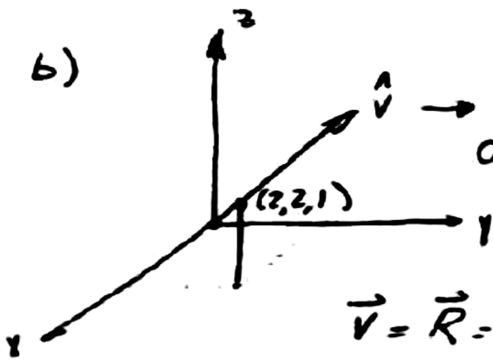
a) the gradient direction is direction of maximum rate of change, $= |\nabla\rho|$

$\nabla\rho = (3x^2y^2z^5 - 2z + 3,\ 2x^3yz^5 + z,\ 5x^3y^2z^4 - 2x + y$

$\nabla\rho|_{(1,2,-1)} = (-7, -5, 20)$

$|\nabla\rho|_{(1,2,-1)}| = \sqrt{7^2+5^2+20^2} = \sqrt{474} \cong 21.77$

b)



$\hat{v} \rightarrow$ vector through pt $(2,2,1)$ directed away from origin

by definition = position vector

$\vec{v} = \vec{R} = (2,2,1)$

$\hat{v} = \dfrac{(2,2,1)}{\sqrt{4+4+1}} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$D_{\hat{v}}\rho = \nabla\rho \cdot \hat{v}$

$\nabla\rho|_{(2,2,1)} = (49, 33, 158)$ (from above)

$D_{\hat{v}}\rho = (49, 33, 158)\cdot\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = \frac{322}{3} \cong 107.33$