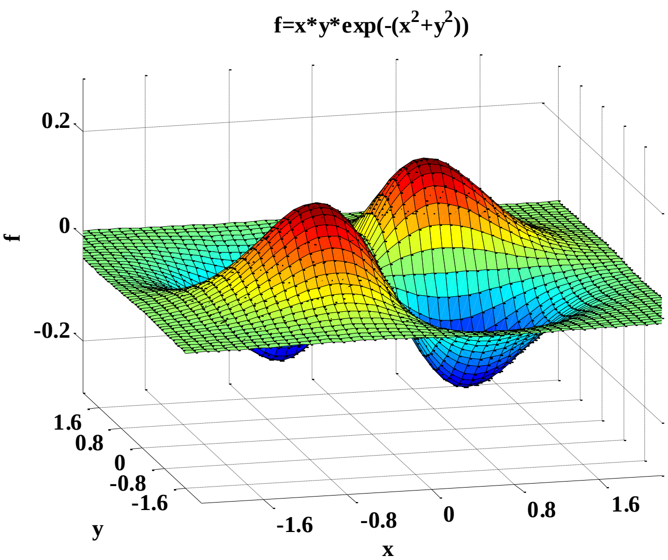
ME 201 TUTORIAL 5 – WINTER 2017

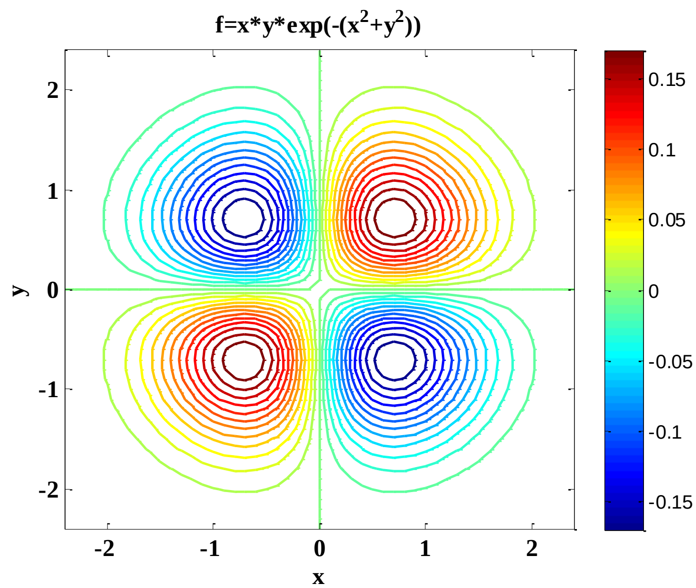
Problem 1 [S. 12.10, Problem 7]

Find all the critical points for $f(x,y) = xye^{-(x^2+y^2)}$ and classify each as yielding a relative maximum, a relative minimum, a saddle point, or none of these.

Solution:

The 3D & contour plots:





For critical points we solve

$$0 = \frac{\partial f}{\partial x} = ye^{-(x^2+y^2)} - 2x^2ye^{-(x^2+y^2)} = y(1-2x^2)e^{-(x^2+y^2)},$$
$$0 = \frac{\partial f}{\partial y} = xe^{-(x^2+y^2)} - 2xy^2e^{-(x^2+y^2)} = x(1-2y^2)e^{-(x^2+y^2)}.$$

Solutions are $(0,0)$, $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ and $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$.

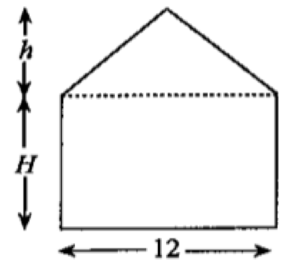
$$\frac{\partial^2 f}{\partial x^2} = y(-4x)e^{-(x^2+y^2)} - 2xy(1-2x^2)e^{-(x^2+y^2)} = 2xy(2x^2-3)e^{-(x^2+y^2)},$$
$$\frac{\partial^2 f}{\partial x \partial y} = (1-2x^2)e^{-(x^2+y^2)} - 2y^2(1-2x^2)e^{-(x^2+y^2)} = (1-2x^2)(1-2y^2)e^{-(x^2+y^2)},$$
$$\frac{\partial^2 f}{\partial y^2} = x(-4y)e^{-(x^2+y^2)} - 2xy(1-2y^2)e^{-(x^2+y^2)} = 2xy(2y^2-3)e^{-(x^2+y^2)},$$
$${f_{xy}}^2 - f_{xx}f_{yy} = [(1-2x^2)^2(1-2y^2)^2 - 4x^2y^2(2x^2-3)(2y^2-3)]e^{-2(x^2+y^2)}.$$

At $(0,0)$, $B^2 - AC = 1$, and therefore $(0,0)$ yields a saddle point.
At $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, $B^2 - AC = -4(1/2)(1/2)(-2)(-2)e^{-2} < 0$, and $A = 2(1/2)(-2)e^{-1} < 0$. These critical points give relative maxima.
At $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$, $B^2 - AC = -4(1/2)(1/2)(-2)(-2)e^{-2} < 0$, and $A = 2(-1/2)(-2)e^{-1} > 0$. These critical points give relative minima.

2. A silo is in the shape of a right-circular cylinder surmounted by a right-circular cone. If the radius of each is 6 m and the total surface area must be 200 m^2. (Not including the base), what heights for the cone and cylinder yield maximum enclosed volume. ?

The volume of the silo is

$$V = \pi(6)^2 H + \frac{1}{3}\pi(6)^2 h = 12\pi(h + 3H).$$



Since area of the silo is $200 = 2\pi(6)H + \pi(6)\sqrt{36 + h^2}$, it follows that

$$V = 12\pi \left[h + 3\left(\frac{200 - 6\pi\sqrt{36 + h^2}}{12\pi} \right) \right]$$

$$= 12\pi h + 600 - 18\pi\sqrt{36 + h^2}, \quad 0 \le h \le \frac{\sqrt{4 \times 10^4 - 36^2\pi^2}}{6\pi}.$$

Here the boundaries of 'h' are used to check the volume obtained by the derivative of V(h) is the maximum
So, h_min <= h <= h_max
h_min = 0, and h_max is obtained by for 200 m^2 area with radius 6 m.
=> 200 = pi*(6)*(36+h_max^2)^0.5

For critical points of V, we solve

$$0 = \frac{dV}{dh} = 12\pi - \frac{18\pi h}{\sqrt{36 + h^2}}.$$

The only positive solution of this equation is $h = 12/\sqrt{5}$. Since

$$V(0) = 600 - 108\pi = 260.7, \quad V\left(\frac{12}{\sqrt{5}}\right) = 347.1, \quad V\left(\frac{\sqrt{4 \times 10^4 - 36^2\pi^2}}{6\pi}\right) = 329.9,$$

V(h_min) ***V(h)*** ***V(h_max)***

V is maximized for $h = 12/\sqrt{5}$ m and $H = (50\sqrt{5} - 27\pi)/(3\sqrt{5}\pi)$ m.

3. A Cobb-Douglas production function has the form $P(x,y) = kx^q y^{1-q}$, where P is the number of items produced per unit time, x is the number of employees, and y is the operating budget for that time. The numbers k>0 and 0<q<1 are constants. Find the least-squares estimates for k and q for the following production data.

Workers, x	100	110	90	100	95	105	110
Budget, y ($)	10000	9000	9000	12000	11000	9500	10000
Production, P	800	810	720	860	810	800	850

If we take logarithms of $P/y = k(x/y)^q$, we obtain $\ln(P/y) = \ln k + q\ln(x/y)$. We now set $R = \ln(P/y)$, $K = \ln k$, and $Z = \ln(x/y)$, then $R = K + qZ$. Least-squares estimates for K and q must satisfy

$$\left(\sum_{i=1}^{7} Z_i^2\right) q + \left(\sum_{i=1}^{7} Z_i\right) K = \sum_{i=1}^{7} Z_i R_i, \qquad \left(\sum_{i=1}^{7} Z_i\right) q + 7K = \sum_{i=1}^{7} R_i.$$

These become

$$147.95q - 32.169K = 81.148, \qquad -32.169q + 7K = -17.643,$$

with solution $q = 0.59388$ and $K = 0.20878$. Consequently, $R = 0.20878 + 0.59388Z$, from which

$$\ln\left(\frac{P}{y}\right) = 0.20878 + 0.59388\ln\left(\frac{x}{y}\right) \implies \frac{P}{y} = e^{0.20878}\left(\frac{x}{y}\right)^{0.59388} \implies P = 1.232x^{0.59388}y^{0.40612}.$$