ME 201 Tutorial 6 - Winter 2018

Problem 1: (Double Integral in Polar Coordinates)

Find the volume inside the ellipsoid

$$x^2 + 4y^2 + z^2 = 64,$$

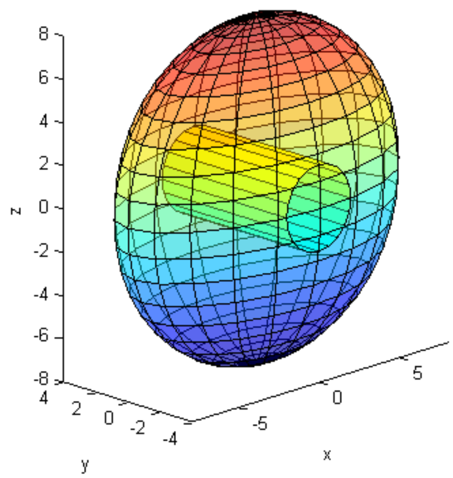
and the cylinder

$$x^2 + z^2 = 4.$$

Solution:

We begin by drawing the shapes (MATLAB script)

```
close all
clear all
[x1,y1,z1]=sphere;
[x2,z2,y2]=cylinder;
figure
hold on
surf(8*x1,4*y1,8*z1)
alpha(0.4)
surf(2*x2,2*sqrt(15)*y2-sqrt(15),2*z2)
axis equal
axis tight
xlabel('x')
ylabel('y')
zlabel('z')
```



(Optional) try Cartesian first:

To use a double integral for volume, the function to be integrated must project onto the bounds of integration, i.e

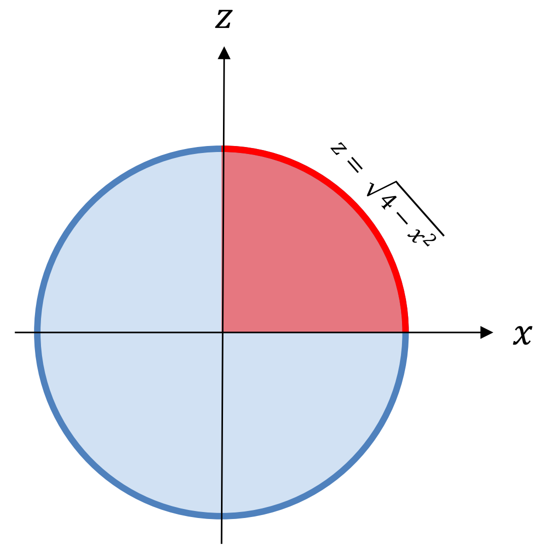
$$h(x,z) = \pm \frac{\sqrt{64 - x^2 - z^2}}{2}$$

Because the volume is symmetric about the xz-plane, we can simply use only the positive side of the height function so long as the volume integral is multiplied by 2 Therefore,

$$V = \iint_A \sqrt{64 - x^2 - z^2}\, dA$$

In order to solve the double integral in Cartesian coordinates we must define boundaries on the x and z axes for the volume projection. Because the projection of the volume is a circle, the order of integration does not matter due to symmetry of all equations. So we choose,

$$z = \sqrt{4 - x^2}$$



This choice is done exploiting the symmetry of the problem so that the projected area on which we are integrating the height function is simply the quarter circle.

Making our bounds on the double integral:

$$4\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{4-x^2}} \sqrt{64 - x^2 - z^2}dzdx$$

<u>Polar Coordinates:</u>

Recall for polar on the xz-plane that:

$$x = r\cos\theta \quad , \quad z = r\sin\theta$$

Therefore the height function over then becomes:

$$h(r,\theta) = \sqrt{64 - r^2}$$

Putting everything back into the integral, keeping in mind that the surface we are integrating the height function is still the first quadrant:

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{2}\sqrt{64 - r^2}rdrd\theta$$

This integral is solved by u substitution or by looking in the Schaum book: $V = 4\pi\left(\frac{256}{3} - 20\sqrt{15}\right)$

Problem 2: [S. 13.8, Prob. 14] Triple Integral in Cartesian Coordinates

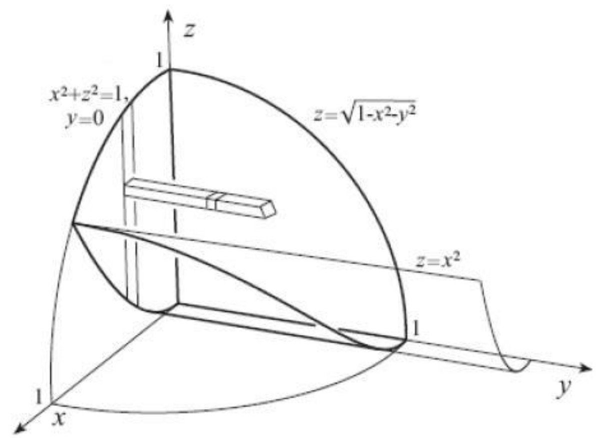
Set up but do not evaluate the following integral

$$\iiint (x^2 + z^2 + y^2)dV$$

Where V is the volume bounded above by $z = (1 - x^2 - y^2)^{\frac{1}{2}}$, and below by $z = x^2$.

Solution:

From the image we see that we can easily simplify by only computing a quarter of the volume we can simplify the integral. We shall choose to integrate in Cartesian coordinates because of the $z = x^2$ surface, which does not lend itself well to cylindrical or spherical coordinate systems.



First we decide upon a good integration order for dV.

Because of the direct relation we have for z and x ($z = x^2$) the integration order most optimal would be:

$$dV = dy\,dz\,dx$$

We draw a line through the volume that is parallel to the y axis. The line enters at $y = 0$, and leaves at $y = (1 - x^2 - z^2)^{\frac{1}{2}}$. Therefore, the y-limits of integration are:

$$\int_0^{(1-x^2-z^2)^{\frac{1}{2}}} x^2 + z^2 + y^2\, dy$$

Now that we have the bounds on y we can look at the volume's shadow projected on the xz-plane. The z-limits of integration are:

$$\int_{x^2}^{(1-x^2)^{\frac{1}{2}}} \int_0^{(1-x^2-z^2)^{\frac{1}{2}}} x^2 + z^2 + y^2\, dy\, dz$$

Finally, the limits on x are determined by the group of all lines L on the x-axis (equate $x^2 = \sqrt{1 - x^2}$ and complete the square). Also, not forgetting the multiplication by 4, the integral is given by:

$$4 \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} \int_{x^2}^{(1-x^2)^{\frac{1}{2}}} \int_0^{(1-x^2-z^2)^{\frac{1}{2}}} (x^2 + z^2 + y^2)\, dy\, dz\, dx$$

Problem 3: [S. 13.12, Prob. 18] Triple Integral in Spherical Coordinates

Evaluate the following triple iterated integral using spherical coordinates

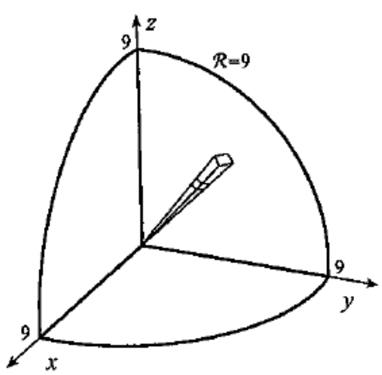
$$\int_0^9 \int_0^{\sqrt{81-y^2}} \int_0^{\sqrt{81-x^2-y^2}} \frac{1}{x^2+y^2+z^2}\, dz\ dx\ dy$$

Solution:

Equation $x^2 + y^2 + z^2 = \rho^2$ defines a sphere with a radius of ρ located at centre $(0,0,0,)$. Therefore in spherical coordinate, $\frac{1}{x^2+y^2+z^2}$ can be replaced by $\frac{1}{\rho^2}$.

Furthermore, by looking at the upper and lower bonds of above integral, we can see that they define the first octant volume inside the sphere $x^2 + y^2 + z^2 = 9^2$. Below the first octant of this sphere is shown.

According to the figure, the limits for the spherical integral are:



$0 \le \rho \le 9$, $0 \le \varphi \le \frac{\pi}{2}$, and $0 \le \theta \le \frac{\pi}{2}$

Therefore, we have:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^9 \frac{1}{\rho^2} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 9 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi\, d\theta$$

$$= 9 \int_0^{\frac{\pi}{2}} \{-\cos\varphi\}_0^{\frac{\pi}{2}} d\theta = 9\{\theta\}_0^{\frac{\pi}{2}} = \frac{9\pi}{2}$$