ME 201 Tutorial #9
Line Integrals & conservative fields

1. Find the work done by the force field $\vec{F}(x,y) = \left(e^y + y e^x\right)\hat{i} + \left(x e^y + e^x\right)\hat{j}$ acting on a particle that moves
   a. from $(a,0)$ to $(-a,0)$ along the x-axis
   b. from $(a,0)$ to $(-a,0)$ along the upper half of the circle $x^2 + y^2 = a^2$
   c. once around the circle $x^2 + y^2 = a^2$

2. Given a scalar (potential) function $\phi = 3x^2 y - y^4 + x^3$ and that $\vec{F} = \nabla\phi$ demonstrate independence of path by calculating the work required to move between $A(0,0)$ and $B(2,4)$ along any path of your choosing.

1) Let $\vec{F}(x,y) = (e^{y} + ye^{x})\hat{i} + (xe^{y} + e^{x})\hat{j}$

Find the work done by the force $\vec{F}$ if it acts on a particle that moves :

a) from $(a,0)$ to $(-a,0)$ along the x-axis

test for independence of path : $\nabla \times \vec{F} = 0$?

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{y} + ye^{x} & xe^{y} + e^{x} & 0 \end{vmatrix} = \hat{i}\left(-\frac{\partial}{\partial z}(xe^{y}+e^{x})\right) - \hat{j}\left(-\frac{\partial}{\partial z}(e^{y} +\right.$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(xe^{y}+e^{x}) - \frac{\partial}{\partial y}(e^{y} + ye\right.$$

$\vec{\nabla} \times \vec{F} = \hat{k}\left(e^{y} + e^{x} - e^{y} - e^{x}\right) = 0$ ✓ $\therefore$ the field is conservative

$\therefore$ $\vec{\nabla}\phi = \vec{F}$ exists, whereby ϕ is a scalar fnc
and thus $\int_C \vec{F} \cdot d\vec{R} = \phi(B) - \phi(A)$

Find ϕ: $\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \vec{\nabla}\phi$

constant of integration

$\therefore$ $\frac{\partial \phi}{\partial x} = e^{y} + ye^{x} \Rightarrow \phi = xe^{y} + ye^{x} + f(y,z)$

$\frac{\partial \phi}{\partial y} = xe^{y} + e^{x} \Rightarrow \phi = xe^{y} + ye^{x} + f(x,z)$

$\frac{\partial \phi}{\partial z} = 0 \Rightarrow \phi = f(x,y)$ $\therefore$ $\underline{f(y,z)} = \underline{f(x,z)} = \underline{0}$

$\phi = xe^{y} + ye^{x}$ and $W = \phi(B) - \phi(A)$

$\phi(B) = \phi(-a,0) = -ae^{0} = -a$ while $\phi(A) = \phi(a,0) = ae^{0} = a$

$\therefore$ $W = -a - a = -2a$

b) from $(a,0)$ to $(-a,0)$ along the upper half of the circle $x^2+y^2=a^2$

- since field is conservative (work independent of path), $\underline{W=-2a}$

c) once around the circle $x^2+y^2=a^2$

- since field is conservative, $\underline{\underline{W=0}}$

2)

$\phi = 3x^2y - y^4 + x^3$ for $\vec{F} = \nabla\phi$

from $A(0,0)$ to $B(2,4)$ → Path #1 $\quad x=t \quad 0 \le t \le 2$
$\quad y=2t$

$\vec{F} = \nabla\phi = (6xy+3x^2)\hat{i} + (3x^2-4y^3)\hat{j}$

$$\int_C \vec{F}\cdot d\vec{R} = \int_0^2 (6t(2t)+3t^2)(1) + (3t^2-32t^3)(2)\,dt$$

$$= \int_0^2 21t^2 - 64t^3\,dt = 7t^3 - 16t^4 \Big|_0^2 = -200$$

→ Path 2 $\quad x=t \quad 0 \le t \le 2$
$\quad y=t^2$

$$\int_C \vec{F}\cdot d\vec{R} = \int_0^2 (6t^3+3t^2) + (3t^2-4t^6)(2t)\,dt$$

$$= \int_0^2 3t^2 + 12t^3 - 8t^7\,dt$$

$$= t^3 + 3t^4 - t^8 \Big|_0^2 = 8 + 48 - 256 = -200$$

Check:

$$\int_C \vec{F}\cdot d\vec{R} = \phi(B) - \phi(A) = 3(4)(4) - (4)^4 + (2)^3 = -200$$

proves path independence