State Equations

Reading	**Problems**
6-4 ⟶ 6-12	

The Thermodynamics of State

IDEAL GAS

The defining equation for a ideal gas is

$$\frac{Pv}{T} = constant = R$$

Knowing that $v = V/m$

$$\frac{PV}{Tm} = constant = R$$

where R is a gas constant for a particular gas (as given in C&B Tables A-1 and A-2).

An Isentropic Process for an Ideal Gas

Given:

- constant specific heats over a wide range of temperature
- $ds = 0$
- $du = c_v dT \equiv c_v = \left(\frac{\partial u}{\partial T}\right)_V$
- $dh = c_p dT \equiv c_p = \left(\frac{\partial h}{\partial T}\right)_P$

Gibb’s equation can be written as

$$Tds = du + Pdv = c_v dT + Pdv = 0 \qquad (1)$$

where $ds = 0$ because we have assumed an isentropic process.

The definition of enthalpy is

$$h = u + Pv$$

Taking the derivative yields

$$dh = \underbrace{du + Pdv}_{\equiv Tds} + vdP$$

$$dh = Tds + vdP \;\; \Rightarrow \;\; Tds = 0 = dh - vdP$$

$$\boxed{c_p dT - vdP = 0} \qquad (2)$$

Equating Eqs. (1) and (2) through the dT term gives

$$\frac{dP}{P} = -\frac{c_p}{c_v}\frac{dv}{v} \qquad (3)$$

Integrating (3) from its initial state to a final state

$$P_1 v_1^k = P_2 v_2^k = constant = Pv^k$$

where

$$k = \frac{c_p}{c_v}$$

The product of $P \cdot v^k$ remains constant for an ideal gas when:

- specific heats are constant

- the gas undergoes an isentropic process ⟶ reversible + adiabatic

Combining this result with the ideal gas equation of state

$$\boxed{\frac{T_2}{T_1} = \left(\frac{v_1}{v_2}\right)^{k-1} = \left(\frac{P_2}{P_1}\right)^{(k-1)/k}}$$

The isentropic process is a special case of a more general process known as a polytropic process

where ⟶ $Pv^n = constant$ and n is any number.

Special Cases

$n = 1$	$Pv = RT = constant$	$\Rightarrow$ isothermal process
$n = 0$	$Pv^0 = constant = P$	$\Rightarrow$ isobaric process (constant pressure)
$n = k$	$Pv^k = constant$	$\Rightarrow$ isentropic process ($k = c_p/c_v$)
$n = \infty$	$Pv^\infty = constant$	$\Rightarrow$ isochoric process (constant volume)

Relative Pressure and Relative Specific Volume

- typically we assume specific heat to be constant with respect to temperature
- *but* when temperature swings are significant, this assumption can lead to inaccuracies, i.e.

$T\ (K)$	$c_p\ (kJ/kg \cdot K)$	**% difference**
300	1.0057	
1000	1.1417	13.5
2500	1.688	67.8

- the relative pressure and relative volume tables (C&B Table A-17), provide an accurate way of including the temperature effects on specific heat for ideal gases during isentropic processes
- note: the specific heat ratio term given by $k = c_p/c_v$ will also be influenced by temperature

- **Procedure:**
  - given T_1, P_1 and P_2 for an isentropic process
  - determine P_{r1} at T_1 from Table A-17
  - calculate P_{r2}, where

$$\left(\frac{P_2}{P_1}\right)_{s=const} = \frac{P_{r2}}{P_{r1}}$$

  - read T_2 from Table A-17 for the calculated value of P_{r2}
- use a similar procedure if volume is known instead of pressure, where

$$\left(\frac{v_2}{v_1}\right)_{s=const} = \frac{v_{r2}}{v_{r1}}$$

In Summary

For an ideal gas with constant c_p and c_v

$$\begin{aligned} Pv &= RT \\ u_2 - u_1 &= c_v(T_2 - T_1) \\ h_2 - h_1 &= c_p(T_2 - T_1) \end{aligned}$$

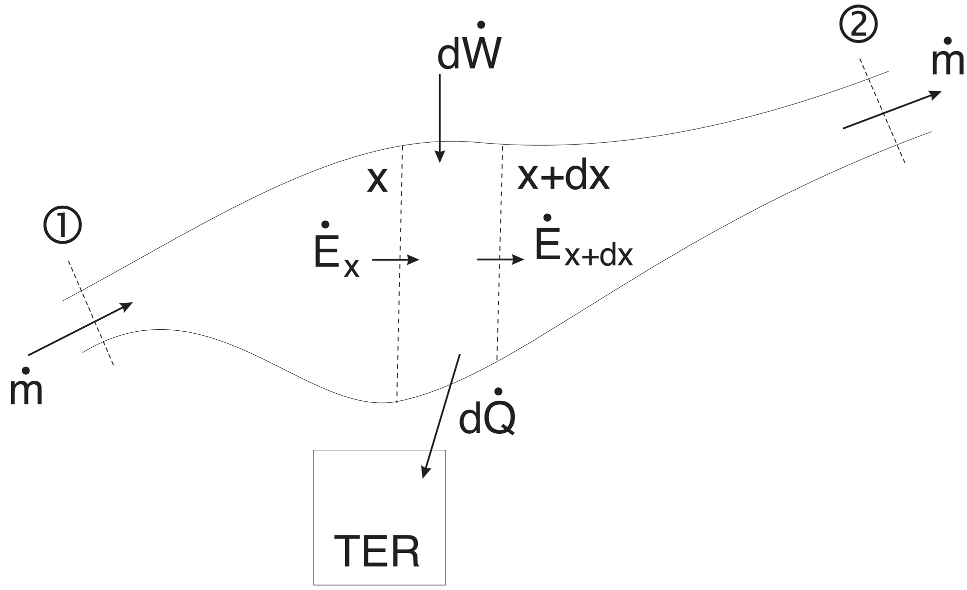
There are 3 forms of a change in entropy as a function of T & v, T & P, and P & v.

$$\begin{aligned} s_2 - s_1 &= c_v \ln\frac{T_2}{T_1} + R\ln\frac{v_2}{v_1} \\ &= c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1} \\ &= c_p \ln\frac{v_2}{v_1} + c_v\ln\frac{P_2}{P_1} \end{aligned}$$

$$R = c_p - c_v$$

A General Formulation

Steady State, Steady Flow in a Flow Channel of Arbitrary Cross-section with Work and Heat Transfer



$$\begin{aligned} d\dot{E} &= \dot{E}_{final} - \dot{E}_{initial} \\ &= \dot{E}_{x+dx} - \dot{E}_{x} \end{aligned}$$

where

$$\begin{aligned} \dot{E} &= \dot{m}(e + Pv) \\ &= \dot{m}\left(u + \frac{(v^*)^2}{2} + gz + Pv\right) \end{aligned}$$

From the 1st law

$$\begin{matrix} rate\ of\ energy \\ storage \end{matrix} = \begin{matrix} rate\ of \\ work \end{matrix} + \begin{matrix} rate\ of \\ heat\ transfer \end{matrix} + \begin{matrix} net\ rate\ of\ energy \\ leaving\ the\ system \end{matrix}$$

$$\frac{dE_{CV}}{dt} = d\dot{W} - d\dot{Q} - d\dot{E} \qquad (1)$$

where $\frac{dE_{CV}}{dt} = 0$ for steady state.

Equation (1) becomes

$$0 = d\dot{W} - d\dot{Q} - \dot{m}\, d\left[u + Pv + \frac{(v^*)^2}{2} + gz\right] \qquad (2)$$

From the 2nd law

$$\begin{matrix} rate\ of \\ entropy \\ storage \end{matrix} = \left\{ \begin{matrix} rate\ of \\ entropy \\ inflow \end{matrix} - \begin{matrix} rate\ of \\ entropy \\ outflow \end{matrix} \right\} + \begin{matrix} rate\ of \\ entropy \\ production \end{matrix}$$

$$\frac{dS_{CV}}{dt} = [\dot{m}s]_x - [\dot{m}s]_{x+dx} - \frac{d\dot{Q}}{T_{TER}} + \dot{\mathcal{P}}_S$$

where $\frac{dS_{CV}}{dt} = 0$ for steady state.

$$0 = -\dot{m}ds - \frac{d\dot{Q}}{T_{TER}} + \dot{\mathcal{P}}_S$$

or

$$d\dot{Q} = T_{TER}\dot{\mathcal{P}}_S - T_{TER}\dot{m}ds \qquad (3)$$

Combining (2) and (3) through $d\dot{Q}$

$$\boxed{T_{TER}\,\dot{\mathcal{P}}_S - T_{TER}\,\dot{m}ds = d\dot{W} - \dot{m}\, d\left(u + Pv + \frac{(v^*)^2}{2} + gz\right)} \qquad (4)$$

Equation (4) can be used for any SS-SF process.

Special Cases

Reversible, SS-SF Process

Reversible implies $\Rightarrow \dot{\mathcal{P}}_S = 0$

- frictionless process
- heat transfer is allowed but must be across $\Delta T \rightarrow 0$
- which means $T_{TER} \approx T_{CV} = T$

Equation 4 becomes

$$\frac{d\dot{W}}{\dot{m}} = -Tds + \underbrace{du + d(Pv)}_{= \underbrace{du + Pdv}_{=Tds} + \underbrace{vdP}} + d\left(\frac{(v^*)^2}{2}\right) + d(gz) \qquad (5)$$

Therefore

$$\frac{d\dot{W}}{\dot{m}} = vdP + d\left(\frac{(v^*)^2}{2}\right) + d(gz) \qquad (6)$$

Integrating Eq. (6) between the inlet and the outlet

$$\frac{\dot{W}}{\dot{m}} = \int_{in}^{out} vdP + \underbrace{\left.\frac{(v^*)^2}{2}\right|_{in}^{out}}_{\Delta KE} + \underbrace{\left. gz\right|_{in}^{out}}_{\Delta PE} \qquad (7)$$

but ΔKE and ΔPE are usually negligible.

If $\Delta KE + \Delta PE = 0$

$$\boxed{\frac{\dot{W}}{\dot{m}} = \int_{in}^{out} vdP} \qquad (8)$$

Equation can be used for a reversible, SS-SF flow in a liquid or a gas.

If we keep in mind

$$\rho_{liq} >> \rho_{gas} \quad \Rightarrow \quad v_{liq} << v_{gas}$$

i.e. water @ $25\ ^\circ C\ \ \rho = 997\ kg/m^3$ and air @ $25\ ^\circ C\ \ \rho = 1.18\ kg/m^3$

Therefore

$$\left(\frac{\dot{W}}{\dot{m}}\right)_{liq} << \left(\frac{\dot{W}}{\dot{m}}\right)_{gas}$$

For example: the work required to operate a pump is much less that that required to operate a compressor.

Incompressible Substance

This is a special case of Eq. (8) where $v = constant = v_{in} - v_{out}$.

From Equation (8)

$$\frac{\dot{W}}{\dot{m}} = v_{in}(P_{out} - P_{in}) \qquad (9)$$

The work term represents the minimum work required to pump a liquid from P_{in} to P_{out} with negligible ΔKE and ΔPE.

Incompressible Substance and $d\dot{W} = 0$

From Eq. (6)

$$vdP + d\left(\frac{(v^*)^2}{2}\right) + d(gz) = 0 \qquad (10)$$

Therefore

$$d\left(\frac{P}{\rho}\right) + d\left(\frac{(v^*)^2}{2}\right) + d(gz) = 0$$

$$d\left(\frac{P}{\rho}+\frac{(v^*)^2}{2}+gz\right)=0 \qquad (11)$$

Integrating gives

$$\frac{P}{\rho}+\frac{(v^*)^2}{2}+gz=constant \qquad (12)$$

Equation (12) is Bernoulli's equation for frictionless flow with constant density. The constant is Bernoulli's constant, which remains constant along a streamline for steady, frictionless, incompressible flow.

Isothermal Ideal Gas, Compression/Expansion

This is a special case of Eq. (8) for an ideal gas where $Pv = RT$

$$Pv = constant = (Pv)_{in} = (Pv)_{out}$$

$$\frac{\dot{W}}{\dot{m}} = \int_{in}^{out} v dP = \int_{in}^{out} (Pv)_{in} \frac{dP}{P}$$

Therefore

$$\boxed{\frac{\dot{W}}{\dot{m}} = P_{in} v_{in} \ln\left(\frac{P_{out}}{P_{in}}\right)} \qquad (13)$$

Isentropic Ideal Gas, Compression/Expansion

Isentropic implies a reversible and adiabatic process where $s = constant$. With an ideal gas, $Pv^k = constant$ and $(Pv^k)_{in} = (Pv^k)_{out}$.

Equation (8) becomes

$$\frac{\dot{W}}{\dot{m}} = \int_{in}^{out} v dP = \int_{in}^{out} \left[\frac{(Pv^k)_{in}}{P}\right]^{1/k} dP$$

$$\boxed{\frac{\dot{W}}{\dot{m}} = \left(\frac{k}{k-1}\right)(Pv)_{in}\left[\left(\frac{P_{out}}{P_{in}}\right)^{(k-1)/k} - 1\right] = c_p(T_{out} - T_{in})} \tag{14}$$

The right side of Eq. (14) is based on the fact that $\Delta KE + \Delta PE = 0$ and $dh = du + dPv$ and $du = 0$. Which leads to $h = \int vdP$.

Note: for the same inlet state and pressure ratio

$$\Rightarrow \left(\frac{\dot{W}}{\dot{m}}\right)_{rev.,isothermal} < \left(\frac{\dot{W}}{\dot{m}}\right)_{rev.,adiabatic}$$