Problem Set for Elliptic Integrals and Functions **Due Date: February 26, 2004**

1. Determine the perimeter of the ellipse

$$4x^2 + 9y^2 = 36$$

2. Obtain the expression for the length of arc of the ellipse

$$x^2 + \frac{y^2}{4} = 1$$

between $(0, 2)$ and $(1/2, \sqrt{3})$. Note that $b > a$ in this problem. Compute the arc length to three decimal places.

3. Compute to four decimal places the following integrals

i) $\displaystyle\int_0^{\pi/4} \frac{dt}{\sqrt{1 - \frac{1}{2}\sin^2 t}}$ ii) $\displaystyle\int_0^{\pi/3} \sqrt{1 - \frac{3}{4}\sin^2 t}\; dt$

iii) $\displaystyle\int_0^{\pi/2} \frac{dt}{\sqrt{1 - \frac{1}{15}\sin^2 t}}$ iv) $\displaystyle\int_0^{\pi/2} \sqrt{1 - \frac{80}{81}\sin^2 t}\; dt$

4. By means of the substitution $x = 2\sin\theta$, show that

$$\int_0^2 \frac{dx}{\sqrt{(4 - x^2)(9 - x^2)}} = \frac{1}{3}\mathrm{K}\left(\frac{2}{3}\right)$$

5. Prove that

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}} = \sqrt{2}\,\mathrm{K}\left(\frac{1}{\sqrt{2}}\right)$$

6. Given $q = 1/2$, compute k, K and K' to six decimal places.

7. Compute to three decimal places the area of the ellipsoid with semi-axes $\mathbf{3, 2,}$ and $\mathbf{1}$.

8. The thermal constriction resistance on an isothermal, elliptical disk $(\boldsymbol{a > b})$ on an insulated isotropic half-space of thermal conductivity $\boldsymbol{k}$ is

$$\boldsymbol{R^* = kaR = \mathbf{K}\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right)}$$

Compute $\boldsymbol{R^*}$ to four decimal places for the following values of $\boldsymbol{a/b : 1, 1.5, 2, 3, 4}$ and $\mathbf{5}$. Use the arithmetic-geometric mean method of Gauss.

9. Derivatives of the elliptic integrals. Show that

i) $$\boldsymbol{\frac{d\mathbf{E}}{dk} = \frac{\mathbf{E} - \mathbf{K}}{k}}$$

ii) $$\boldsymbol{\frac{d^2\mathbf{E}}{dk^2} = -\frac{1}{k}\frac{d\mathbf{K}}{dk} = -\frac{\mathbf{E} - (k')^2\mathbf{K}}{k^2(k')^2}}$$

10. Integrals of the elliptic integrals. Show that

i) $$\boldsymbol{\int \mathbf{K}\, dk = \frac{\pi k}{2}\left\{1 + \sum_{n=1}^{\infty} \frac{[(2n)!]^2 k^{2n}}{(2n+1)2^{4n}(n!)^4}\right\}}$$

ii) $$\boldsymbol{\int k\mathbf{K}\mathbf{L}\, dk = \mathbf{E} - (k')^2\mathbf{K}}$$

iii) $$\boldsymbol{\int k\, \mathbf{E}\, dk = \frac{1}{3}[(1 + k^2)\mathbf{E} - (k')^2\mathbf{K}]}$$

11. Show that

i) $$\boldsymbol{\int_0^1 \frac{k\mathbf{E}\, dk}{k'} = \int_0^1 \mathbf{E}(u')\, du = \frac{\pi^2}{8}}$$

ii) $$\boldsymbol{\int_0^1 \frac{\mathbf{K}\, dk}{1 + k} = \frac{\pi^2}{8}}$$

12. Derivatives of elliptic integrals with respect to the argument

i) $$\boldsymbol{\frac{d}{d\phi}\,\mathbf{F}(\phi,k) = \frac{1}{\sqrt{1-k^2\sin^2\phi}}}$$

ii) $$\boldsymbol{\frac{d}{d\phi}\,\mathbf{E}(\phi,k) = \sqrt{1-k^2\sin^2\phi}}$$