Problem Set for Error and Complementary Error Function

Due Date: February 12, 2004

1. Evaluate the following integrals to four decimal places using either power series, asymptotic series or polynomial approximations:

   a) $\displaystyle\int_0^2 e^{-x^2}\,dx$ b) $\displaystyle\int_{0.001}^{0.002} e^{-x^2}\,dx$

   c) $\displaystyle\frac{2}{\sqrt{\pi}}\int_{1.5}^{\infty} e^{-x^2}\,dx$ d) $\displaystyle\frac{2}{\sqrt{\pi}}\int_{5}^{10} e^{-x^2}\,dx$

   e) $\displaystyle\int_1^{1.5}\left(\frac{1}{2}e^{-x^2}\right)\,dx$ f) $\displaystyle\sqrt{\frac{2}{\pi}}\int_1^{\infty}\left(\frac{1}{2}e^{-x^2}\right)\,dx$

2. The value of **erf 2** is **0.995** to three decimal places. Compare the number of terms required in calculating this value using:

   a) the convergent power series, and
   b) the divergent asymptotic series.

   Compare the approximate errors in each case after two terms; after ten terms.

3. For the function $\mathbf{ierfc}(x)$ compute to four decimal places when $x = \mathbf{0},\ \mathbf{0.2},\ \mathbf{0.4},\ \mathbf{0.8},$ and $\mathbf{1.6}$.

4. Prove that

   i) $\sqrt{\pi}\,\mathrm{erf}(x) = \gamma\left(\frac{1}{2}, x^2\right)$

   ii) $\sqrt{\pi}\,\mathrm{erfc}(x) = \Gamma\left(\frac{1}{2}, x^2\right)$

   where $\gamma\left(\frac{1}{2}, x^2\right)$ and $\Gamma\left(\frac{1}{2}, x^2\right)$ are the incomplete Gamma functions defined as:

$$\gamma(a, y) = \int_0^y e^{-u}u^{a-1}\,du$$

   and

$$\Gamma(a, y) = \int_y^{\infty} e^{-u}u^{a-1}\,du$$

5. Show that $\theta(x,t) = \theta_0 \text{ erfc}(x/2\sqrt{\alpha t})$ is the solution of the following diffusion problem:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \qquad\qquad x \geq 0,\ t > 0$$

and

$$\begin{aligned} \theta(0,t) &= \theta_0, \quad \text{constant} \\ \theta(x,t) &\to 0 \quad \text{as} \quad x \to \infty \end{aligned}$$

6. Given $\theta(x,t) = \theta_0 \text{ erf } x/2\sqrt{\alpha t}$:

   i) Obtain expressions for $\dfrac{\partial\theta}{\partial t}$ and $\dfrac{\partial\theta}{\partial x}$ at any x and all $t > 0$

   ii) For the function $\dfrac{\sqrt{\pi}}{2}\dfrac{x}{\theta_0}\dfrac{\partial\theta}{\partial x}$

   show that it has a maximum value when $x/2\sqrt{\alpha t} = 1/\sqrt{2}$ and the maximum value is $1/\sqrt{2e}$.

7. Given the transient point source solution valid within an isotropic half space

$$T = \frac{q}{2\pi k r}\text{ erfc}(r/2\sqrt{\alpha t}), \quad dA = r\ dr\ d\theta$$

derive the expression for the transient temperature rise at the centroid of a circular area (πa^2) which is subjected to a uniform and constant heat flux q. Superposition of point source solutions allows one to write

$$T_0 = \int_0^a \int_0^{2\pi} T\ dA$$

8. For a dimensionless time $Fo < 0.2$ the temperature distribution within an infinite plate $-L \leq x \leq L$ is given approximately by

$$\frac{T(\zeta, Fo) - T_s}{T_0 - T_s} = 1 - \left\{\text{erfc}\ \frac{1-\zeta}{2\sqrt{Fo}} + \text{erfc}\ \frac{1+\zeta}{2\sqrt{Fo}}\right\}$$

for $0 \leq \zeta \leq 1$ where $\zeta = x/L$ and $Fo = \alpha t/L^2$.

Obtain the expression for the mean temperature $(\overline{T}(Fo) - T_s)/(T_0 - T_s)$ where

$$\overline{T} = \int_0^1 T(\zeta, Fo)\, d\zeta$$

The initial and surface plate temperature are denoted by T_0 and T_s, respectively.

9. Compare the approximate short time ($Fo < 0.2$) solution:

$$\theta(\zeta, Fo) = 1 - \sum_{n=1}^{3} (-1)^{n+1} \left\{ \text{erfc} \frac{(2n-1) - \zeta}{2\sqrt{Fo}} + \text{erfc} \frac{(2n-1) + \zeta}{2\sqrt{Fo}} \right\}$$

and the approximate long time ($Fo > 0.2$) solution

$$\theta(\zeta, Fo) = \sum_{n=1}^{3} \frac{2(-1)^{n+1}}{\delta_n}\, e^{-\delta_n^2 Fo}\, \cos(\delta_n \zeta)$$

with $\delta_n = (2n-1)\pi/2$.

For the centerline ($\zeta = 0$) compute to four decimal places $\theta(0, Fo)_{ST}$ and $\theta(0, Fo)_{LT}$ for $Fo = 0.02, 0.06, 0.1, 0.4, 1.0$ and 2.0 and compare your values with the "exact" values given in Table 1.

Table 1: Exact values of $\theta(0, Fo)$ for the Infinite Plate

Fo	$\theta(0, Fo)$
0.02	1.0000
0.06	0.9922
0.10	0.9493
0.40	0.4745
1.0	0.1080
2.0	0.0092