Problem Set for Gamma and Beta Functions **Due Date: January 29, 2004**

1. Use the definition of the gamma function with a suitable change of variable to prove that

   i) $$\int_0^\infty e^{-ax}x^n\,dx = \frac{1}{a^{n+1}}\,\Gamma(n+1) \qquad \text{with } n > -1,\ a > 0$$

   ii) $$\int_a^\infty \exp(2ax - x^2)\,dx = \frac{\sqrt{\pi}}{2}\,\exp(a^2)$$

2. Prove that

$$\int_0^{\pi/2} \sin^n\theta\,d\theta = \int_0^{\pi/2} \cos^n\theta\,d\theta = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma([1+n]/2)}{\Gamma([2+n]/2)}$$

3. Show that

$$\Gamma\left(\frac{1}{2}+x\right)\ \Gamma\left(\frac{1}{2}-x\right) = \frac{\pi}{\cos\pi x}$$

   Plot your results over the range $-10 \le x \le 10$.

4. Evaluate $\Gamma\left(-\frac{1}{2}\right)$ and $\Gamma\left(-\frac{7}{2}\right)$.

5. Show that the area enclosed by the axes $x = 0,\ y = 0$ and the curve $x^4 + y^4 = 1$ is

$$\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{8\sqrt{\pi}}$$

   Use both the Dirichlet integral and a conventional integration procedure to substantiate this result.

6. Express each of the following integrals in terms of the gamma and beta functions and simplify when possible.

i) $\displaystyle\int_0^1 \left(\frac{1}{x} - 1\right)^{1/4} dx$

ii) $\displaystyle\int_a^b (b-x)^{m-1}(x-a)^{n-1}\, dx,$ with $b > a,\ m > 0,\ n > 0$

iii) $\displaystyle\int_0^\infty \frac{dt}{(1+t)\sqrt{t}}$

Note: Validate your results using various solution procedures where possible.

7. Compute to 5 decimal places

$$\frac{A}{4ab} = \frac{1}{2n}\,\frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{\Gamma\left(\frac{2}{n}\right)}$$

for $n = 0.2,\ 0.4,\ 0.8,\ 1.0,\ 2.0,\ 4.0,\ 8.0,\ 16.0,\ 32.0,\ 64.0,\ 100.0$

8. Sketch $x^3 + y^3 = 8$. Derive expressions of the integrals and evaluate them in terms of Beta functions for the following quantities:

a) the first quadrant area bounded by the curve and two axes
b) the centroid $(\overline{x}, \overline{y})$ of this area
c) the volume generated when the area is revolved about the $y-$axis
d) the moment of inertia of this volume about its axis

Note: Validate your results using various solution procedures where possible.

9. Starting with

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}\, dt}{\sqrt{t}}$$

and the transformation $y^2 = t$ or $x^2 = t$, show that

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty \exp\left[-(x^2+y^2)\right]\, dx\, dy$$

Further prove that the above double integral over the first quadrant when evaluated using polar coordinates (r, θ) yields

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$