Problem Set on Legendre, Hermite, Laguerre and Chebyshev Polynomials
Due Date: April 12, 2004

1. Obtain the Legendre polynomial $P_4(x)$ from Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n\right]$$

2. Obtain the Legendre polynomial $P_4(x)$ directly from Legendre's equation of order 4 by assuming a polynomial of degree 4, i.e.

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

3. Obtain the Legendre polynomial $P_6(x)$ by application of the recurrence formula

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

assuming that $P_4(x)$ and $P_5(x)$ are known.

4. Obtain the Legendre polynomial $P_2(x)$ from Laplace's integral formula

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x + \sqrt{x^2-1}\cos t)^n \, dt$$

5. Find the first three coefficients in the expansion of the function

$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$$

in a series of Legendre polynomials $P_n(x)$ over the interval $(-1, 1)$.

6. Find the first three coefficients in the expansion of the function

$$f(\theta) = \begin{cases} \cos\theta & 0 \le \theta \le \pi/2 \\ 0 & \pi/2 \le \theta \le \pi \end{cases}$$

in a series of the form

$$f(\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta) \qquad\qquad 0 \leq \theta \leq \pi$$

7. Obtain the associated Legendre functions $P_2^1(x),\ P_3^2(x)$ and $P_2^3(x)$.

8. Verify that the associated Legendre function $P_3^2(x)$ is a solution of Legendre's associated equation for $m = 2,\ n = 3$.

9. Verify the result

$$\int_{-1}^{1} P_n^m(x)\ P_k^m(x)\ dx = 0 \qquad\qquad n \neq k$$

for the associated Legendre functions $P_2^1(x)$ and $P_3^1(x)$.

10. Verify the result

$$\int_{-1}^{1} \left[P_n^m(x)\right]^2\ dx = \frac{2}{2n+1}\ \frac{(n+m)!}{(n-m)!}$$

for the associated Legendre function $P_1^1(x)$.

11. Obtain the Legendre functions of the second kind $Q_0(x)$ and $Q_1(x)$ by means of

$$Q_n(x) = P_n(x) \int \frac{dx}{[P_n(x)]^2(1-x^2)}$$

12. Obtain the function $Q_3(x)$ by means of the appropriate recurrence formula assuming that $Q_0(x)$ and $Q_1(x)$ are known.

13. Obtain the first three Hermite polynomials $H_0(x),\ H_1(x)$ and $H_2(x)$ by means of the corresponding Rodrigue's formula.

14. By means of the generating function obtain the Hermite polynomials $H_0(x),\ H_1(x)$ and $H_2(x)$.

15. Show that $\boldsymbol{H_3(x)}$ satisfies the Hermite differential equation of order $\mathbf{3}$.

16. Show that

$$\int_{-\infty}^{\infty} e^{-x^2} [H_2(x)]^2 dx = 8\sqrt{\pi}$$

17. Expand the function $f(x) = x^3 - 3x^2 + 2x$ in terms of Hermite polynomials such that

$$f(x) = \sum_{n=0}^{\infty} A_n H_n(x)$$

18. Evaluate

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} H_n(x)\ dx$$

19. Obtain the first three Laguerre polynomials $L_0(x),\ L_1(x)$ and $L_2(x)$ by means of the corresponding Rodrigue's formula.

20. By means of the generating function obtain the Laguerre polynomials $L_0(x),\ L_1(x),$ and $L_2(x)$.

21. Show that $L_2(x)$ satisfies the Laguerre differential equation of order $\mathbf{2}$.

22. By means of the appropriate recurrence formula obtain $L_3(x)$ assuming that $L_0(x)$ and $L_1(x)$ are known.

23. Expand the function $f(x) = x^3 - 3x^2 + 2x$ in terms of Laguerre polynomials such that

$$f(x) = \sum_{n=0}^{\infty} A_n L_n(x)$$

24. Obtain the first three Chebyshev polynomials $T_0(x),\ T_1(x)$ and $T_2(x)$ by means of the Rodrigue's formula.

25. Show that the Chebyshev polynomial $T_3(x)$ is a solution of Chebyshev's equation of order 3.

26. By means of the recurrence formula obtain Chebyshev polynomials $T_2(x)$ and $T_3(x)$ given $T_0(x)$ and $T_1(x)$.

27. Show that $T_n(1) = 1$ and $T_n(-1) = (-1)^n$

28. Show that $T_n(0) = 0$ if n is odd and $(-1)^{n/2}$ if n is even.

29. Setting $x = \cos\theta$ show that

$$T_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1 - x^2}\right)^n + \left(x - i\sqrt{1 - x^2}\right)^n\right]$$

where $i = \sqrt{-1}$.

30. Find the general solution of Chebyshev's equation for $n = 0$.

31. Obtain a series expansion for $f(x) = x^2$ in terms of Chebyshev polynomials $T_n(x)$,

$$x^2 = \sum_{n=0}^{3} A_n T_n(x)$$

32. Express x^4 as a sum of Chebyshev polynomials of the first kind.