303FES99.TEX

UNIVERSITY OF WATERLOO

Department of Mechanical Engineering
ME 303 Advanced Engineering Mathematics

Spring Term 1999 — August 4, 1999
M.M. Yovanovich — 9:00–12:00 Noon

Open book examination. Aids are allowed such as course text, lecture material and any material from the ME 303 Web site, calculator and Spiegel's Mathematical Handbook. All questions must be answered and they are of equal value.

Read each problem carefully before beginning the analysis. Answer all questions which are asked.

Show all steps and state clearly all assumptions made. Material which is not legible will not be considered.

The problems deal with one of the following PDEs: Laplace, Poisson, Diffusion and Wave Equations which are classified as elliptic, parabolic and hyperbolic types. The solution methods employed are of the type: separation of variables, Laplace transform, and similarity transformation.

The mathematical problems can come from several engineering areas such as conduction heat transfer, mass transfer, fluid mechanics, and dynamics of solids.

Good luck.

Problem 1. (20)

A wire of constant cross-sectional area A moves along the positive $x-$axis with constant velocity V. The thermophysical properties: mass density ρ, specific heat capacity c_p, thermal conductivity k, and thermal diffusivity $\alpha = k/(\rho\, c_p)$ are assumed to be constant. The temperature excess is defined as $\theta(x,t) = T(x,t) - T_\infty$ where T_∞ is the constant ambient (sink) temperature. An energy balance on an appropriate differential control volume $d\mathcal{V} = A\, dx$, followed by division by the arbitrary volume $d\mathcal{V}$ leads to the following partial differential equation (PDE):

$$k\frac{\partial^2\theta}{\partial x^2} + \rho c_p V\frac{\partial\theta}{\partial x} - \frac{hP}{A}\theta = \rho c_p \frac{\partial\theta}{\partial t}, \quad t > 0, \quad 0 < x < L$$

where h is the heat transfer coefficient and P is the perimeter of the wire. The terms of the PDE from left to right are respectively the conduction term, the advection term, the heat loss term, and the energy storage term.

(a) Divide through by k, then introduce the parameters $m^2 = (hP)/(kA)$ and α, and rewrite the PDE.

(b) What are the units of V/α and m^2 which appear in the new PDE?

(c) Substitute $\theta(x,t) = X(x)T(t)$ into the PDE, and separate it into two related ODEs which are appropriate for this problem.

(d) Obtain the solutions for $T(t)$ and $X(x)$.

(e) Consider the special case where the wire is stationary, i.e. $V = 0$, and the wire surface is adiabatic, i.e. $h = 0$, except for the two end surfaces at $x = 0$ and $x = L$. Obtain the $X(x)$ solution for the boundary conditions: $\theta_x(0,t) = 0$ and $\theta(L,t) = 0$ with initial condition $\theta(x,0) = \theta_i = T_i - T_\infty$. Clearly identify the *eigenfunctions* and the *eigenvalues*.

Problem 2. (20)

Consider steady, fully-developed laminar flow of a fluid in a long rectangular duct: $-a \leq x \leq a$, $-b \leq y \leq b$. The governing partial differential equation (PDE) for the fluid velocity $u(x,y)$ is the two-dimensional Poisson equation:

$$u_{xx} + u_{yy} = \frac{1}{\mu}\frac{\partial P}{\partial z} = \text{constant}$$

where μ is the fluid viscosity and $\partial P/\partial z$ is the constant pressure drop along the length of the duct.

Taking advantage of symmetry about the axes: $x = 0, y = 0$, the problem can be formulated in the first quadrant: $0 \leq x \leq a$, $0 \leq y \leq b$. The boundary conditions for the first quadrant are

$$x = 0, \quad u_x(0,y) = 0 \quad \text{and} \quad x = a, \quad u(a,y) = 0$$

and

$$y = 0, \quad u_y(x,0) = 0 \quad \text{and} \quad y = b, \quad u(x,b) = 0$$

The boundary conditions on the axes $x = 0$ and $y = 0$ are the zero shear conditions, and on the walls of the duct $x = a, y = b$, the boundary conditions are the no-slip conditions.

(a) Introduce the following solution into the given PDE

$$u(x,y) = v(x) + w(x,y)$$

and separate it into an ODE for $v(x)$ and another homogeneous PDE for $w(x,y)$.

(b) Obtain the solution for the ODE.

(c) Specify the boundary conditions for the PDE for $w(x,y)$.

(d) Specify the ODE for $X(x)$ and the boundary conditions. Obtain its solution.

Problem 3. (20)

An elastic, constant cross-sectional bar of length L is fixed at one end $x = 0$ and it has a concentrated mass M attached to the end at $x = L$. The mass of the bar is $M_{bar} = \rho AL$. The longitudinal displacement $u(x,t)$ of the bar from its equilibrium position is described by the one-dimensional wave equation:

$$u_{xx} = \frac{1}{c^2}u_{tt}, \quad t > 0, \quad 0 < x < L$$

where the constant of the system is defined as

$$c^2 = \frac{E}{\rho}$$

$E =$ Young's modulus and $\rho =$ mass density.

(a) What are the units of the parameter c?

(b) What is the form of the solution of the wave equation, i.e, $u(x,t) = X(x)T(t) =?$

(c) The boundary condition of the fixed end is $u(0,t) = 0$, and the dynamic force in the bar at the free end $x = L$ is equal to the inertia force of the concentrated mass, i.e.

$$AE\left(\frac{\partial u}{\partial x}\right)_{x=L} = -M\left(\frac{\partial^2 u}{\partial t^2}\right)_{x=L}$$

Derive the relation:

$$M_{bar}\cos\delta_n = M\delta_n\sin\delta_n$$

where $\delta_n = \lambda_n L$.

(d) When $M_{bar}/M \to \infty$, what is the relation for δ_n and what are the values of δ_n?

(e) When $M_{bar}/M \to 0$, what is the relation for δ_n and what are the values of δ_n?

(f) Calculate the value of the first root δ_1 for $M_{bar}/M = 1$. Six digit accuracy is acceptable.

Problem 4. (20)

Fick's Second Law of Diffusion describes transient diffusion of atoms through a substance. The one-dimensional diffusion equation is

$$c_{xx} = \frac{1}{\mathcal{D}} u_t, \quad t > 0, \quad x > 0$$

where $c(x,t)$ is the concentration of the diffusing atoms, and $\mathcal{D}$ is the diffusion coefficient assumed to be constant. The initial concentration is $c(x,0) = c_0$ for $x \geq 0$. The boundary conditions for $t > 0$ are:

$$u(0,t) = c_s, \quad \text{and} \quad u(x,t) \to c_0 \quad \text{as} \quad x \to \infty$$

(a) Define the similarity parameter η for the given PDE.

(b) Transform the PDE into a second-order ODE, and specify the boundary conditions.

(c) Obtain the solution of the ODE, and show that a solution of the given PDE has the form (given in your Materials Science Text):

$$\frac{c_s - c_x}{c_s - c_0} = erf\left(\frac{x}{2\sqrt{\mathcal{D}t}}\right), \quad t > 0, \quad x \geq 0$$

where c_s is the concentration of the diffusing atoms at location x below the surface after time t. The special function *erf* is called the error function which can be computed approximately by the relations presented in the course lecture.

(d) For Carbon in FCC iron, the diffusion coefficient is given by the relation:

$$\mathcal{D} = \mathcal{D}_0 \exp\left(\frac{-Q}{RT}\right)$$

where $\mathcal{D}_0 = 0.23 \times 10^{-4}\, m^2/s$, $Q = 137700\, J/mol$ and $R = 8.31\, J/(mol \cdot K)$, and T is the absolute temperature of the iron during the carburising.

Calculate the diffusion time t for $T = 1000°C$, and

$$c_0 = 0.1\%\, C, \quad c_s = 1.2\%\, C, \quad c_x = 0.40\%\, C, \quad \text{for a depth of } 2.2\, mm$$

Problem 5. (20)

A system having volume V and surface area A consists of a solid with constant thermophysical properties: mass density ρ and specific heat capacity c_p. The surface of the system is subjected to a uniform and constant incident heat flux q_i, and there is heat loss by convective cooling from the surface through a constant heat transfer coefficient h. The temperature of the system $T(t)$ is the solution of the ordinary differential equation (ODE):

$$\rho c_p V \frac{dT(t)}{dt} = q_i A - hA\left[T(t) - T_\infty\right], \quad t > 0$$

where T_∞ is the constant ambient temperature. The initial temperature is $T(0) = T_i$.

(a) The transient temperature depends on nine independent parameters:
$T = T(t, T_i, T_\infty, q_i, h, \rho, c_p, A, V)$. For convenience, in the subsequent analysis, let

$$m = \frac{hA}{\rho c_p V} \quad \text{and} \quad n = \frac{q_i A}{\rho c_p V}$$

What are the units of the parameters: m, n and n/m.

(b) Obtain the Laplace transform of the ODE and a relation for $\bar{T}(s)$ where

$$\bar{T}(s) = \mathcal{L}\left\{T(t)\right\}$$

is a function of the parameters: (T_i, T_∞, m, n, s). What are the restrictions on the transform parameter s?

(c) Obtain the solution of the ODE by taking the inverse Laplace transform of $\bar{T}(s)$, i.e.,

$$T(t) = \mathcal{L}^{-1}\left\{\bar{T}(s)\right\}$$

(d) What is the steady-state solution $T(\infty)$?

(e) Put the transient solution $T(t)$ into a dimensionless form such that

$$\phi(\tau) = e^{-\tau}, \quad \tau \geq 0$$

where ϕ and τ are the dimensionless temperature and time respectively.
