UNIVERSITY OF WATERLOO
DEPARTMENT OF MECHANICAL ENGINEERING

ME 305 PARTIAL DIFFERENTIAL EQUATIONS

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Open book final examination. All aids are allowed. All questions must be answered and they are of equal value. Show all steps and state clearly all assumptions made. *Material which is illegible will not be considered.*

The problems deal with Laplace, Diffusion and Wave Equations which are classified as elliptic, parabolic and hyperbolic types. The solution methods employed are separation of variables, Laplace transform and similarity transformation.

The mathematical problems come from several engineering areas such as: conduction heat transfer, mass transfer, fluid mechanics, and dynamics of solids.

Good luck.

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1. Steady-state convective heat transfer from an isothermal plate which is maintained at $T = T_0$ into a laminar flowing fluid at $T = T_\infty$ $(T_0 > T_\infty)$ is approximated by the following partial differential equation (*parabolic type*):

$$\rho c_p u_e \frac{\partial \phi}{\partial x} = k \frac{\partial^2 \phi}{\partial y^2} \quad x \geq 0 \quad y \geq 0$$

The physical parameters: ρ (mass density), c_p (specific heat), and k (thermal conductivity) are assumed to be constants. The dimensionless temperature ϕ is defined as $\phi = (T - T_\infty)/(T_0 - T_\infty)$ where $T = T(x, y)$. The effective fluid velocity u_e is assumed to be constant.

(a) Check the units of the left and right hand sides of the given **PDE**. The units of k are $W/(m \cdot K)$.

(b) By means of dimensional analysis of the **PDE**, define the similarity parameter η. Let $\alpha = k/(\rho c_p)$ in the similarity parameter.

(c) Use the similarity parameter of part (b) above, to transform the given **PDE** into an **ODE** such that ϕ depends on η only, and $0 \leq \eta < \infty$.

(d) Obtain the solution to the **ODE** for the following boundary conditions:

$$\phi(0) = 1 \qquad\qquad \phi(\infty) = 0$$

2. It is found empirically that the transport of neutrons in a reactor core is a diffusion process similar to heat conduction. In particular, the neutron density $u(x, y, z, t)$ (neutrons per unit volume) at any point within the core is the solution of the partial differential equation:

$$\frac{1}{k}\frac{\partial u}{\partial t} = \nabla^2 u + B^2 u$$

The constant k represents the transport of neutrons and the constant B is the so-called *buckling* constant which is determined empirically for each specific mixture of fissionable and moderating material. Since the second term on the right-hand side of the **PDE** accounts for the production of neutrons within the core, it is called the source term.

(a) What are the units of ∇^2, k and B? The space and time units are m and s respectively.

(b) Show that the substitution $u(x, y, z, t) = e^{B^2kt}v(x, y, z, t)$ transforms the $u-$equation into the source-free diffusion equation:

$$\frac{1}{k}\frac{\partial v}{\partial t} = \nabla^2 v$$

Assume that the reactor core can be modeled as a very large *slab* of thickness L. Then the neutron density $u(x, t)$ and the new function $v(x, t)$ are dependent on one space variable and the time. The three-dimensional source and source-free **PDEs** reduce to

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + B^2 u, \quad 0 \le x \le L, \quad t > 0$$

and

$$\frac{1}{k}\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad 0 \le x \le L, \quad t > 0$$

(c) Obtain the neutron density solution, $u(x, t)$, for the following boundary conditions:

$$v(0, t) = v(L, t) = 0$$

and the arbitrary initial condition:

$$v(x, 0) = f(x), \quad 0 \le x \le L$$

3. Use Separation of Variables Method **(SVM)** to obtain the the solution to the following two-dimensional Laplace (*elliptic type*) **PDE** within the rectangular domain domain $\{0 \le x \le a, 0 \le y \le b\}$:

$$\textbf{PDE}: \qquad u_{xx} + u_{yy} = 0$$

with homogeneous *Neumann* and *Robin* conditions along three boundaries:

$$1) \qquad u_x(0, y) = 0$$

$$2) \qquad u_x(a, y) = -\frac{h}{k}\, u(a, y)$$

$$3) \qquad u_y(x, b) = 0$$

and mixed *Neumann* boundary conditions along the boundary $y = 0$:

$$4a) \qquad u_y(x, 0) = 0, \qquad 0 \le x < c$$

$$4b) \qquad u_y(x, 0) = -\frac{q_0}{k} \qquad c < x \le a$$

The physical parameters h, k, q_0 are positive constants.

4. Given the nonhomogeneous, second-order, partial differential equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{\nu}\frac{\partial u}{\partial t} + \frac{1}{\mu}\frac{\partial P}{\partial z} \qquad t > 0, \quad 0 \le r \le a, \quad 0 \le z \le L$$

where the physical parameters: μ, ν and $\partial P/\partial z$ are constants with respect to the independent variables: r and t. The length L of the circular tube of radius a is much larger than the radius, i.e. $L >> a$.

(a) Separate the given nonhomogeneous **PDE** into a homogeneous **PDE** in $w(r, t)$ and a nonhomogeneous **ODE** in $v(r)$ by setting $u(r, t) = w(r, t) + v(r)$.

(b) Separate the homogeneous **PDE** into *three sets* of related **ODEs** by setting $w(r, t) = R(r) \cdot \tau(t)$ where $R(r)$ and $\tau(t)$ are independent space and time functions.

(c) Obtain the appropriate solution $u(r, t)$ for the boundedness condition along the axis, i.e. $u(0, t)$ is finite, the homogeneous *Dirichlet* condition, $u(a, t) = 0$, on the boundary $r = a$, and the initial condition $u(r, 0) = 0$.

5. A fluid having kinematic viscosity ν is contained between two *infinitely* large plates separated by a distance L. Initially the fluid velocity is zero everywhere, i.e. $u(y,0) = 0$, where y is the coordinate perpendicular to the plates, and its origin is located in the lower plate.

Suddenly the upper plate is set into motion such that its velocity $u(L,t) = u_0$, constant.

The partial differential equation which is applicable for the above problem is the homogeneous diffusion (*parabolic type*) equation:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\nu}\frac{\partial u}{\partial t}, \qquad t > 0, \qquad 0 \leq y \leq L$$

The initial condition (**IC**) and boundary conditions (**BCs**) are:

$$u(y,0) = 0, \qquad u(0,t) = 0, \qquad u(L,t) = u_0$$

(a) Nondimensionalize the **PDE**, the **IC** and the **BCs**.

(b) Use the Laplace Transform Method to obtain the solution $\overline{u}(y,s)$.

(c) The shear at the wall $y = L$ is defined as $\tau = \mu \dfrac{\partial u(L,t)}{\partial y}$ where μ is the fluid viscosity. Find the wall shear in the $s-$domain from the $\bar{u}(y,s)$ solution.

(d) Obtain the physical solution $u(y,t)$ by means of *Laplace Transform Tables*.