UNIVERSITY OF WATERLOO
DEPARTMENT OF MECHANICAL ENGINEERING
ME 305 PARTIAL DIFFERENTIAL EQUATIONS

SPRING 1996 June 13, 1996
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Open book examination. All aids are allowed. All questions must be answered and they are of equal value. Show all steps and state clearly all assumptions made. Material which is not legible will not be considered. Good luck.

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1) The steady-state temperature $T(x, y)$ at any point inside a constant cross-section solid whose length in the $z-$direction is very long is given as

$$T(x, y) = T_0 + C\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]$$

where T_0, a, b, C are constants. The parameters: a and b are respectively the semimajor and semiminor axis of the elliptical cross-section.

(a) What are the S.I. units of a, b, C?

(b) It is stated that the above expression is the solution of the Poisson equation

$$\nabla^2 T = -\frac{S}{k}$$

where S represents the uniformly distributed volumetric heat sources whose units are W/m^3, and k is the thermal conductivity whose units are $W/m \cdot K$. Demonstrate by substitution whether the expression satisfies the partial differential equation. If the Poisson equation is satisfied, what is the relationship between the constant C and the source strength S, the thermal conductivity k, and the constants a, b?

(c) The above expression satisfies the nonhomogeneous Dirichlet boundary condition $T(x, y) = T_0$ at all points on the boundary of the ellipse, and $T(x, y) > T_0$ everywhere inside the cross-section due to the volumetric heat sources. Suppose that in consistent units: $T_0 = 0, a = 2, b = 1, S = 250, k = 10$. Sketch the temperature distributions along the $x-$ and $y-$axes. Sketch the isotherms within the cross-section.

(d) Apply Fourier's rate equation: $\vec{q} = -k\nabla T$ to find the magnitude of the heat flux vector at the two points: $x = a, y = 0$ and $x = 0, y = b$. Leave the result in symbolic form.

The heat flux vector and the vector operators in Cartesian coordinates are: $\vec{q} = \vec{i}q_x + \vec{j}q_y$, $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The unit vectors along the $x-$ and $y-$axes are: $\vec{i}$ and $\vec{j}$ respectively.

2) Obtain the solution to the following problem:

(a) Use the *separation of variables method* to obtain the solution to the following partial differential equation **PDE**:

$$\frac{\partial^2 u}{\partial x \partial t} + \sin t = 0, \qquad t > 0$$

given the following initial and boundary conditions:

$$u(x, 0) = x \qquad \text{and} \qquad u(0, t) = 0$$

(b) Verify your answer by *direct* integration and substitution of the initial and boundary conditions.

3) The following linear second-order nonhomogeneous partial differential equation is the governing equation for non-steady heat conduction with convection cooling in a very long thin wire in which there are uniformly distributed heat sources (due to Ohmic heating):

$$au_{xx} - bu_x - cu + d = eu_t, \qquad t > 0 \qquad x \geq 0$$

The wire moves along the positive $x-$axis with a constant velocity U. The constants: a, b, c, d, e are thermophysical parameters of the system. The units of x, t, u and d are respectively: m, s, K and W/m^3.

(a) Determine the units of a, b, c and e.

(b) If $b = \rho c_p U$ where the units of ρ are kg/m^3, determine the units of c_p.

(c) If $b = c = d = e = 0$ in the above equation, what does it reduce to and obtain its solution.

(d) If $b = c = e = 0$ in the above equation, what does it reduce to and obtain its solution.

(e) If $b = d = e = 0$ in the above equation, what does it reduce to and obtain its solution.

(f) If $b = c = d = 0$ in the above equation, what does it reduce to? Is the **PDE** *elliptic type, hyperbolic type* or *parabolic type*? Use the parameter $B^2 - 4AC$ to determine the type. What are the units of the parameter e/a? What combination of time dependent $\tau(t)$ and space dependent $X(x)$ functions satisfy the diffusion equation?