Solutions for Midterm Examination, June 13, 1996

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Problem 1.

Given the steady-state temperature:

$$T(x,y) = T_0 + C\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]$$

(a) What are the units of a, b, C in S.I. units? The units of the parameters are $a[m]$ and $b[m]$, and $C[K]$.

The temperature $T(x,y)$ is a solution of the Poisson equation:

$$\nabla^2 T = -\frac{S}{k}$$

where the parameter $S[W/m^3]$ represents the volumetric sources and $k[W/m \cdot K]$ is the thermal conductivity. Show that $T(x,y)$ satisfies the PDE and find relationship for C.
Taking partial derivatives we get:

$$T_x = C\left[-\frac{2x}{a^2}\right], \qquad T_{xx} = \frac{-2C}{a^2}$$

and

$$T_y = C\left[-\frac{2y}{b^2}\right], \qquad T_{yy} = \frac{-2C}{b^2}$$

Substitution into two-dimensional Poisson equation gives:

$$-\frac{2C}{a^2} - \frac{2C}{b^2} = -\frac{S}{k}$$

Solve for C to get relationship:

$$C = \frac{S/k}{[2/a^2 + 2/b^2]}$$

(c) Given $T_0 = 0, a = 2, b = 1, S = 250, k = 10$ in consistent S.I. units, sketch the temperature distribution along $x-$ and $y-$ axes. Find value of C.

$$C = \frac{250/10}{[2/2^2 + 2/1^2]} = 10$$

Temperature distribution becomes:

$$T(x, y) = 10\left[1 - \frac{x^2}{4} - \frac{y^2}{1}\right]$$

The isotherms (constant values of temperature) are ellipses. Temperatures along the $x-$ and $y-$ axes are:

$$T(x, 0) = 10\left[1 - \frac{x^2}{4}\right]$$

and

$$T(0, y) = 10\left[1 - y^2\right]$$

Because of symmetry, show the temperature plots in the first quadrant only. Apply Fourier's rate equation: $\vec{q} = -k\nabla T$ to find magnitude of heat flux vector at the two points: $x = a, y = 0$ and $x = 0, y = b$ From

$$\mathbf{i}q_x + \mathbf{j}q_y = -k\left[\mathbf{i}\frac{\partial T}{\partial x} + \mathbf{j}\frac{\partial T}{\partial y}\right]$$

we get

$$q_x = -k\frac{\partial T}{\partial x} = -k\left[-\frac{2C}{a^2}\right] = \frac{2kC}{a^2} = 50\left[\frac{W}{m^2}\right]$$

and

$$q_y = -k\frac{\partial T}{\partial y} = -k\left[-\frac{2C}{b^2}\right] = \frac{2kC}{b^2} = 200\left[\frac{W}{m^2}\right]$$

where

$$C = \frac{S/k}{[2/a^2 + 2/b^2]}$$

Problem 2.

Given the problem:

$$\frac{\partial^2 u}{\partial x \partial t} + \sin t = 0, \qquad t > 0$$

with initial and boundary conditions:

$$u(x,0) = x, \qquad u(0,t) = 0$$

(a) Use Separation of Variables Method to get the solution.

Assume $u(x,t) = X(x)\tau(t)$ and substitute into PDE. Taking partial derivatives we get:

$$\frac{\partial u}{\partial t} = X\tau'$$

and

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(X\tau') = X'\tau'$$

Therefore

$$X'\tau' + \sin(t) = 0$$

Divide by τ' to get the separated equation:

$$X' + \frac{\sin(t)}{\tau'} = 0$$

To continue further set the two parts to: (i)$0, 0$; (ii) $\lambda^2, -\lambda^2$; and (iii)$-\lambda^2, \lambda^2$. The first option does not work. The second option gives the separated ODEs with solutions:

$$\tau' = -\frac{1}{\lambda^2}\sin t, \qquad \tau(t) = \frac{1}{\lambda^2}\cos t + C_1$$

$$X' = \lambda^2, \qquad X(x) = \lambda^2 x + C_2$$

The general solution is

$$u(x,t) = X(x)\tau(t) = \left[\lambda^2 x + C_2\right]\left[C_1 + \frac{1}{\lambda^2}\cos t\right]$$

Apply the initial and boundary conditions to get:

$$u(x,0) = \left[\lambda^2 x + C_2\right]\left[C_1 + \frac{1}{\lambda^2}\right] = x$$

and

$$u(0,t) = C_2 \left[C_1 + \frac{1}{\lambda^2} \cos t \right] = 0$$

Solving for the constants of integration we find that $C_1 = C_2 = 0$, and the solution is

$$u(x,t) = x \cos t$$

The third option gives the separated ODEs with solutions:

$$\tau' = \frac{1}{\lambda^2} \sin t, \qquad \tau(t) = -\frac{1}{\lambda^2} \cos t + C_1$$

$$X' = -\lambda^2, \qquad X(x) = -\lambda^2 x + C_2$$

The general solution is

$$u(x,t) = X(x)\tau(t) = \left[-\lambda^2 x + C_2 \right] \left[C_1 - \frac{1}{\lambda^2} \cos t \right]$$

Apply the initial and boundary conditions to get:

$$u(x,0) = \left[\lambda^2 x + C_2 \right] \left[C_1 - \frac{1}{\lambda^2} \right] = x$$

and

$$u(0,t) = C_2 \left[C_1 - \frac{1}{\lambda^2} \cos t \right] = 0$$

Solving for the constants of integration we find that $C_1 = C_2 = 0$, and the solution is identical to the solution for option (ii):

$$u(x,t) = x \cos t$$

(b) Verify the result by direct integration and application of initial and boundary conditions.

Integration with respect to t gives:

$$\frac{\partial u}{\partial x} = \cos t + C_1$$

Second integration with respect to x gives:

$$u(x,t) = x \cos t + C_1 x + C_2$$

Use initial and boundary conditions:

$$u(x,0) = x + C_1 x + C_2 = x, \qquad u(0,t) = C_2 = 0$$

Therefore $C_1 = 0$ and $C_2 = 0$, which gives the solution

$$\boxed{u(x,t) = x\cos t}$$

as determined above.

Problem 3.

Give the PDE

$$au_{xx} - bu_x - cu + d = eu_t, \quad t > 0, \quad x > 0$$

and units:

$$x[m], \quad t[s], \quad u[K], \quad d[W/m^3]$$

(a) Find units of parameters: a, b, c and e.
Units of each term must be identical to units of parameter $d[W/m^3]$. Therefore we can find units in following manner:

$$a[units]\left[\frac{K}{m^2}\right] = \left[\frac{W}{m^3}\right] \Longrightarrow a\left[\frac{W}{mK}\right]$$

$$b[units]\left[\frac{K}{m}\right] = \left[\frac{W}{m^3}\right] \Longrightarrow b\left[\frac{W}{m^2K}\right]$$

$$c[units]\,[K] = \left[\frac{W}{m^3}\right] \Longrightarrow c\left[\frac{W}{m^3K}\right]$$

$$e[units]\left[\frac{K}{s}\right] = \left[\frac{W}{m^3}\right] \Longrightarrow e\left[\frac{Ws}{m^3K}\right] \quad \text{or} \quad e\left[\frac{J}{m^3K}\right]$$

(b) Given $b = \rho c_p U$ with units of velocity $U[m/s]$ and mass density $\rho[kg/m^3]$ find units of c_p.

$$b\left[\frac{W}{m^2K}\right] = \rho\left[\frac{kg}{m^3}\right] c_p\,[units]\,U\left[\frac{m}{s}\right] \Longrightarrow c_p\left[\frac{Ws}{kgK}\right] \quad \text{or} \quad c_p\left[\frac{J}{kgK}\right]$$

(c) Given $b = c = d = e = 0$ in above PDE, what does it reduce to and what is the solution?
PDE becomes:

$$u_{xx} = 0, \quad x > 0$$

with general solution:

$$u(x) = C_1 x + C_2$$

(d) Given $b = c = e = 0$ in above PDE, what does it reduce to and what is the solution?
PDE becomes:

$$a u_{xx} + d = 0 \quad \text{or} \quad u_{xx} = -\frac{d}{a}$$

with general solution:

$$u(x) = -\frac{d}{a}\frac{x^2}{2} + C_1 x + C_2$$

(e) Given $b = d = e = 0$ in above PDE, what does it reduce to and what is the solution?
PDE becomes:

$$a u_{xx} - c u = 0 \quad \text{or} \quad u_{xx} - \frac{c}{a} u = 0$$

with general solution:

$$u(x) = C_1 \exp\left(\sqrt{\frac{c}{a}} x\right) + C_2 \exp\left(-\sqrt{\frac{c}{a}} x\right)$$

or the equivalent general solution is

$$u(x) = C_1 \cosh\left(\sqrt{\frac{c}{a}} x\right) + C_2 \sinh\left(\sqrt{\frac{c}{a}} x\right)$$

(f) Given $b = c = d = 0$ in above PDE, what does it reduce to. Is the PDE elliptic, hyperbolic or parabolic type? Use the parameter $B^2 - 4AC$ to determine type. What are the units of e/a? What combination of time $\tau(t)$ and space $X(x)$ functions satsify the diffusion equation? The PDE becomes:

$$a u_{xx} = e u_t \quad \text{or} \quad u_{xx} = \frac{e}{a} u_t, \quad t > 0$$

Compare new PDE against general equation:

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

With $x \leftrightarrow x$ and $y \leftrightarrow t$ we find:

$$A = 1, \quad B = 0, \quad C = 0, \quad D = 0, \quad E = \frac{e}{a}, \quad F = 0, \quad G = 0$$

Therefore

$$B^2 = 4AC = 0^2 - (4)(1)(0) = 0 - 0 = 0$$

The new PDE is parabolic type.
Units of e/a are found from:

$$\frac{e}{a}\left[\frac{Ws}{m^3K}\right]\left[\frac{mK}{W}\right] \Longrightarrow \frac{e}{a}\left[\frac{s}{m^2}\right]$$

What combination of independent time$\tau(t)$ and space $X(x)$ functions satisfy the diffusion equation?
Let $u(x,t) = X(x)\tau(t)$ and substitute into the PDE and separate the variables. We get:

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{\tau'}{\tau}$$

where for convenience set $\alpha = a/e$. The equality must hold for all values of x and t. This can be satisfied by setting the lhs and the rhs equal to the same constant. There are three options: (i) 0, (ii) - λ^2, (iii) λ^2. The three options give the following combinations which satisfy the PDE:

$$u(x,t) = X(x)\tau(t) = (C_1 x + C_2)C_3$$

$$u(x,t) = X(x)\tau(t) = (C_1\cos(\lambda x) + C_2\sin(\lambda x))\,C_3\exp(-\lambda^2\alpha t)$$
$$u(x,t) = X(x)\tau(t) = (C_1\cosh(\lambda x) + C_2\sinh(\lambda x))\,C_3\exp(\lambda^2\alpha t)$$

or equivalently

$$u(x,t) = X(x)\tau(t) = (C_1\exp(\lambda x) + C_2\exp(-\lambda x))\,C_3\exp(\lambda^2\alpha t)$$

Note that the third option leads to a solution which becomes unbounded as $t \to \infty$.