303MTS98.TEX

UNIVERSITY OF WATERLOO

Department of Mechanical Engineering
ME 303 Advanced Engineering Mathematics

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Open book examination. All aids are allowed. All questions must be answered and they are of equal value. Show all steps and state clearly all assumptions made. Material which is not legible will not be considered. Good luck.

Problem 1.

A viscometer is a device used to determine the viscosity of fluids such as engine oils. The device consists of an inner cylinder of diameter D_i which is placed inside another cylinder of inner diameter $D_o > D_i$. The annular space between the two cylinders of thickness $H = (D_o - D_i)/2$ is filled with a fluid whose viscosity μ is determine by means of Newton's Law of Viscosity:

$$\tau_w = -\mu \frac{du}{dy}$$

where τ_w is the measured wall shear during steady rotation of the inner cylinder while the outer cylinder is held fixed, and du/dy is the velocity gradient, and $u(y)$ is the steady-state velocity distribution of the fluid in the narrow annular space where $H << D_i$. Since the annular space thickness is very small relative to the cylinder diameters, the fluid motion is modeled as fluid confined by two infinite parallel plates with a separation distance H. The partial differential equation which describes the transient fluid motion $u(y,t)$ is

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\nu}\frac{\partial u}{\partial t}, \qquad t > 0, \qquad 0 < y < H$$

where $\nu = \mu/\rho$ is the kinematic viscosity. The initial velocity within the space is $u(y,0) = 0$, and the no-slip boundary conditions require $u(0,t) = U$ on the inner rotating cylinder and $u(H,t) = 0$ on the fixed outer cylinder.

(a) What are the S.I. units of the transport parameter ν if the units of the parameters are: $u[m/s], y[m], t[s]$?

(b) Find the steady-state velocity distribution.

(c) By means of the steady-state velocity distribution and Newton's Law of Viscosity obtain the relationship between τ_w and μ, U, H.

(d) If the units of the wall shear are $\tau_w[N/m^2]$, what are the units of μ?

(e) Let $u(y,t) = v(y) + w(y,t)$ and show that when the Separation of Variables Method (SVM) is used, i.e. assuming $w(y,t) = Y(y)T(t)$, the following Sturm-Liouville Problem (SLP) arises:

$$Y'' + \lambda^2 Y = 0, \qquad 0 \le y \le H$$

with homogeneous Dirichlet boundary conditions:

$$Y(0) = 0, \qquad Y(H) = 0$$

(f) What eigenfunctions and eigenvalues satisfy the SLP?

Problem 2.

You are given the following dimensional, nonhomogeneous heat equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{P}{k} = \frac{1}{\alpha}\frac{\partial u}{\partial t}, \qquad t > 0, \qquad 0 < x < L$$

with initial condition: $u(x,0) = u_0$, and homogeneous Neumann and Dirichlet boundary conditions:

$$\frac{\partial u(0,t)}{\partial x} = 0, \qquad u(L,t) = 0$$

(a) Nondimensionalize the partial differential equation with the dimensionless position: ζ, dimensionless time: τ, and dimensionless temperature: ϕ with

$$\zeta = \frac{x}{L}, \qquad \tau = \frac{\alpha t}{L^2}, \qquad \phi = \frac{u}{u_{\text{ref}}}$$

where u_{ref} is an arbitrary reference temperature to be determined below.

(b) Select the arbitrary reference temperature so that the dimensionless source term becomes 1. Now nondimensionalize the initial and boundary conditions.

(c) Specify the nondimensional steady-state ODE and the boundary conditions, and obtain the solution.

(d) From the dimensionless steady-state solution determine the dimensionless heat transfer rates at the two boundaries $\zeta = 0$ and $\zeta = 1$ defined as:

$$Q_0^\star = -\frac{d\phi(0)}{d\zeta}, \qquad Q_1^\star = -\frac{d\phi(1)}{d\zeta}$$

Problem 3.

Consider the conformal map defined by the complex exponential function:

$$u(x,y) + iv(x,y) = e^{(ix-y)} = e^{ix}e^{-y} = e^{-y}\left(\cos x + i \sin x\right)$$

with $i = \sqrt{-1}$.

(a) Equating the real and imaginary parts show that

$$u(x,y) = e^{-y}\cos x, \qquad v(x,y) = e^{-y}\sin x$$

satisfy the Laplace equations:

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

(b) Obtain the Fourier coefficients for the function:

$$f(x) = \frac{x}{L}\left(1 - \frac{x}{L}\right), \qquad 0 \leq x \leq L$$

(c) Write the first three terms of the Fourier series expansion of the given function.

The following integrals and identities may be helpful:

$$\int x\cos(ax)dx = \frac{1}{a^2}\left[\cos(ax) + ax\sin(ax)\right]$$

$$\int x\sin(ax)dx = \frac{1}{a^2}\left[\sin(ax) - ax\cos(ax)\right]$$

$$\int x^2\cos(ax)dx = \frac{1}{a^3}\left[2ax\cos(ax) + (a^2x^2 - 2)\sin(ax)\right]$$

$$\int x^2\sin(ax)dx = \frac{1}{a^3}\left[2ax\sin(ax) - (a^2x^2 - 2)\cos(ax)\right]$$

For integer values of n,

$$\sin(n\pi) = 0, \qquad \cos(n\pi) = (-1)^n$$