ME 303 M.M. Yovanovich

Week 2

Lecture 1

• Hand out Problem Set 1.

• ODEs in cartesian, polar and spherical coordinates; TAs will discuss some solutions in the tutorials. Discuss how to obtain solution of homogeneous ODE in spherical coordinates. Use Maple to obtain the solution.

• Boundary Conditions (BCs). There are three types:
(i) BC of the First Kind or Dirichlet BC. u is specified at points on the boundary.
(ii) BC of the Second Kind or Neumann BC. $\partial u/\partial n$, gradient of u normal to boundary is specifed at points on the boundary.
(iii) BC of the Third Kind or Robin BC. Linear combination of Dirichlet and Neumann BCs is specified at points on the boundary: $a\dfrac{\partial u}{\partial n} + bu = 0$; where the parameters: a, b are positive physical parameters.

Lecture 2

• Makeup Lecture 1.

• Error and complementary error functions. See Spiegel Handbook, $35.1 - 35.6$, Page 183.
• Definitions

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} d\beta \quad \text{and} \quad erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\beta^2} d\beta$$

where β is a dummy variable.

• Relations and Properties.

$$erf(x) + erfc(x) = 1$$

$$erf(0) = 0, \quad erfc(0) = 1$$

$$erf(\infty) = 1, \quad erfc(\infty) = 0$$

$$erf(-x) = erf(x)$$

$$\frac{d\,erf(x)}{dx} = \frac{2}{\sqrt{\pi}}e^{-x^2} \quad \text{and} \quad \frac{d\,erfc(x)}{dx} = -\frac{2}{\sqrt{\pi}}e^{-x^2}$$

Demonstrate that the function: $u = C erf(\eta)$ where C is an arbitrary constant satisfies the ODE:

$$\frac{d^2u}{d\eta^2} + 2\eta\frac{du}{d\eta} = 0$$

This is an important second-order ODE.

• How to generate PDEs. See Chapter 12, first section. We will not spend much time on this topic. Demonstrate that given $u(x, y) = yf(x)$ where $f(x)$ is an arbitrary function of x, elimination of $f(x)$ gives the first-order PDE: $u = y\dfrac{\partial u}{\partial y}$.

• Also demonstrate that $u(x, y) = f(x + y) + g(x - y)$ satisfies the PDE: $u_{xx} = u_{yy}$, where $f(x+y)$ and $g(x-y)$ are arbitrary functions. Here are two examples:

$$u(x, y) = \sin(x + y) + e^{x-y}$$

and

$$u(x, y) = (x + y)^3 + \tan(x - y)$$

• Separation of Variables Method (SVM). See Problem 7 of Problem Set 1. There are several examples given.

• Separate the 2-D Laplace equation in cartesian coordinates: $u_{xx} + u_{yy} = 0$ into three sets of independent ODEs. Let $u(x, y) = X(x)Y(y)$ where $X(x)$ and $Y(y)$ are independent functions and substitute into the PDE.

$$\frac{\partial u}{\partial x} = \frac{\partial(XY)}{\partial x} = \frac{\partial(X)}{\partial x}Y + X\frac{\partial(Y)}{\partial x} = X'Y$$

because $\partial Y/\partial x = 0$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial(X'Y)}{\partial x} = \frac{\partial(X')}{\partial x}Y + X'\frac{\partial(Y)}{\partial x} = X''Y$$

Similarly we find

$$\frac{\partial^2 u}{\partial y^2} = XY''$$

The given PDE becomes

$$X''Y + XY'' = 0$$

Next we divide through by the assumed solution XY which gives separated relation:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

The numerator and denominator of the first term depends on x only. Also the numerator and denominator of the second term depends on y only. Demonstrate that both terms are constants by taking the derive with respect to x and then with respect to y. The two terms must be constants. There are three options for the constants:

$$\begin{array}{rcc} i) & 0, & 0 \\ ii) & -\lambda^2, & +\lambda^2 \\ iii) & -\lambda^2, & +\lambda^2 \end{array}$$

where the parameter $\lambda > 0$ is the separation constant. Each of these options will give two independent ODEs. They are respectively

$$i) \quad X'' = 0 \quad \text{and} \quad Y'' = 0,$$

$$ii) \quad X'' + \lambda^2 X = 0 \quad \text{and} \quad Y'' - \lambda^2 Y = 0,$$

$$iii) \quad X'' - \lambda^2 X = 0 \quad \text{and} \quad Y'' + \lambda^2 Y = 0,$$

These are second-order ODEs with constant coefficients.

Lecture 3

- 1-D Diffusion Equation. $u(x,t)$.

$$u_{xx} = \frac{1}{\alpha} u_t, \qquad t > 0, \qquad 0 < x < L$$

where $\alpha > 0$ is a thermophysical paramter. Let $u(x,t) = X(x)T(t)$. Substitute into the PDE to get

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{T'}{T}$$

The LHS and RHS must be equal to the same constant. There are three options:

$$\begin{array}{rcc} i) & 0, & 0 \\ ii) & -\lambda^2, & -\lambda^2 \\ iii) & +\lambda^2, & +\lambda^2 \end{array}$$

The three options lead to the three sets of independent separated ODEs:

$$i)\ X'' = 0 \qquad \text{and} \qquad T' = 0$$

$$ii)\ X'' + \lambda^2 X = 0 \qquad \text{and} \qquad T' + \lambda^2 \alpha T = 0$$

$$iii)\ X'' - \lambda^2 X = 0 \qquad \text{and} \qquad T' - \lambda^2 \alpha T = 0$$

The three sets of solutions are

$$i)\ X(x) = C_1 x + C_2 \qquad \text{and} \qquad T(t) = C_3$$

$$ii)\ X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \qquad \text{and} \qquad T(t) = C_3 e^{-\lambda^2 \alpha t}$$

$$iii)\ X(x) = C_1 \cosh \lambda x + C_2 \sinh \lambda x \qquad \text{and} \qquad T(t) = C_3 e^{\lambda^2 \alpha t}$$

The first solution is independent of time. The second solution has a negative exponential in time which goes to zero for very large times, and the third solution has a positive exponential in time which becomes unbounded at very large times. The second solution is

$$u(x,t) = X(x)T(t) = C_3 e^{-\lambda^2 \alpha t} \left[C_1 \cos \lambda x + C_2 \sin \lambda x \right]$$

Discuss heating and cooling problems which are identical when the temperature difference $\theta(t)$ is introduced. For both problems the temperature difference $\theta(t)$ is initially $\theta(0) = \theta_i > 0$, and $\theta(t) \to 0$ as $t \to \infty$. The form of the solution must be of the form

$$\frac{\theta(t)}{\theta_i} = e^{-\text{constant } t}, \qquad t > 0$$

The constant will be found later.

• Use SVM on the 1-D Wave Equation where $u(x,t)$ to find the three sets of separated ODEs. The 1-D wave equation is

$$u_{xx} = \frac{1}{c^2} u_{tt}, \qquad t > 0, \qquad 0 < x < L$$

where $c > 0$. The time equations will be second-order in time.

Lecture 4

• Classification of Linear Second-Order Partial Differential Equations. The two-dimensional PDE where $u(x,y)$ has the general form:

$$\boxed{Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G}$$

where the coefficients $A, B, C, D, E, F,$ and G are functions of x and y or they can be constants. The PDEs can be classified as:

$$1. \quad \text{Hyperbolic if} \quad B^2 - 4AC > 0$$

$$2. \quad \text{Parabolic if} \quad B^2 - 4AC = 0$$

$$3. \quad \text{Elliptic if} \quad B^2 - 4AC < 0$$

• The PDEs are defined to be *homogeneous* if $G = 0$, otherwise they are defined to be *nonhomogeneous*.

• The PDEs are defined to be *nonlinear* if they contain terms like

$$u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial u^2}{\partial x}$$

• Examples of Hyperbolic, Parabolic and Elliptic types.

• Diffusion equation: $u_{xx} = u_t$.
Here we have $x = x, t = y$. Comparing given equation with general equation shows that: $A = 1, B = 0, C = 0, D = 0, E = -1, F = 0, G = 0$. Therefore $B^2 - 4BC = (0)^2 - (4) \times (0) \times (0) = 0$. For all values of x and t the diffusion equation is parabolic type.

• Wave equation: $u_{xx} = u_{tt}$.
Here we have $x = x, t = y$. Comparing given equation with general equation shows that: $A = 1, B = 0, C = -1, D = 0, E = 0, F = 0, G = 0$. Therefore $B^2 - 4BC = (0)^2 - (4) \times (1) \times (-1) = 4 > 0$. For all values of x and t the wave equation is hyperbolic type.

• Laplace equation: $u_{xx} + u_{yy} = 0$.
Here we have $x = x, y = y$. Comparing given equation with general equation shows that: $A = 1, B = 0, C = 1, D = 0, E = 0, F = 0, G = 0$. Therefore $B^2 - 4BC = (0)^2 - (4) \times (1) \times (1) = -4 < 0$. For all values of x and t the Laplace equation is elliptic type.

• Wave Equation Appears in Several Physical Problems such as:

• 1. Vibrating Strings and Membranes (rectangular, circular)
• 2. Transverse Vibrations of Beams
• 3. Longitudinal Vibrations of Elastic Bars

- 4. Torsional Vibrations of Elastic Rods
- 5. Sound Waves in Tubes or Pipes
- 6. Transmission of Electricity Along an Insulated, Low-Resistance Cable
- 7. Long Water Waves in a Straight Canal
- 8. Linearized Supersonic Air Flow
- 9. Many other examples from physics and engineering

- Vibrating String With Several Forces.

- String length is L. One end is fixed at $x = 0$ and the other end is fixed at $x = L$. The tension in the string is T and its linear mass density is ρ. The displacement of the string from its equilibrium position is $y(x,t)$. The displacements occur in the $xy-$plane, and the displacements are small, i.e. $|y(x,t)| << L$. The local slope of the string is also small, therefore $dy/dx << 1$.

- Forces acting on the element of arclength ds.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \approx dx$$

- Tension Forces

At x, the vertical component of the tension force

$$T \ \sin\theta \approx T \ \tan\theta = T \ \frac{\partial y}{\partial x}$$

acts in the negative $y-$direction.
At $x + dx$, the vertical component of the tension force is using the Taylor series approximation:

$$T \ \frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\left(T \ \frac{\partial y}{\partial x}\right) dx$$

acts in the positive $y-$direction. The net vertical component of the tension force is

$$\frac{d}{dx}\left(T \ \frac{dy}{dx}\right) dx$$

acting in the positive $y-$direction.

- Body Force due to gravity is $\rho g dx$ acting in the negative $y-$direction.

- Friction damping force in negative $y-$direction is related to the string velocity; it is $k_1 \dfrac{\partial y}{\partial t} dx$.

• Retarding force in negative $y-$direction is related to the string displacement from equilibrium; it is $k_2 y dx$.

• External Periodic Force is $F\ \cos(\omega t)dx$.

• Newton's Law of Motion applied to the string element gives:

$$\sum \text{Forces}_{\text{y direction}} = (\text{mass}) \times (\text{acceleration})$$

Therefore

$$\frac{\partial}{\partial x}\left(T\ \frac{\partial y}{\partial x}\right)dx + F\ \cos(\omega t)dx - k_1\frac{\partial y}{\partial t}dx - k_2 y dx - \rho g dx = \rho\frac{\partial^2 y}{\partial t^2}dx$$

Dividing through by the mass of the element ρdx and assuming that the tension is constant, we obtain the general wave equation for a string:

$$c^2\frac{\partial^2 y}{\partial x^2} + \frac{F}{\rho}\cos(\omega t) - \frac{k_1}{\rho}\frac{\partial y}{\partial t} - \frac{k_2}{\rho}y - g = \frac{\partial^2 y}{\partial t^2}, \qquad t > 0, \qquad 0 < x < L$$

where $c^2 = T/\rho$ and $c = \sqrt{T/\rho}$ is the wave propagation velocity.

• Free Vibrations of a String.
Here we set $F = 0, k_1 = 0, k_2 = 0$ and ignore the gravitational force. The general wave equation becomes

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}, \qquad t > 0, \qquad 0 < x < L$$

or

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \qquad t > 0, \qquad 0 < x < L$$

or

$$y_{xx} = \frac{1}{c^2}y_{tt} \qquad t > 0, \qquad 0 < x < L$$

This linear homogeneous PDE will be solved for a number of initial conditions. At $t = 0$, the initial displacement from equilibrium is $y(x, 0) = f(x)$, and the initial velocity is $\dfrac{\partial y(x,0)}{\partial t} = g(x)$ where both $f(x)$ and $g(x)$ are arbitrary functions.