ME 303 M.M. Yovanovich

Week 5

Lecture 1

• Sturm-Liouville Problem (SLP) Cartesian Coordinates.

• $u = u(x, y)$ or $u = u(x, t)$.

• Partial differential equations.

$$\text{2D Laplace Equation:} \qquad u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H$$

$$\text{1D Diffusion Equation:} \qquad u_{xx} = \frac{1}{\alpha} u_t, \quad t > 0, \quad 0 < x < L$$

$$\text{1D Wave Equation:} \qquad u_{xx} = \frac{1}{c^2} u_{tt}, \quad t > 0, \quad 0 < x < L$$

where α and c^2 are positive constants.

• Separation of Variables Method (SVM) is used to obtain two sets of independent ODES. Let $u(x, y) = X(x)Y(y)$ and let $u(x, t) = X(x)T(t)$. One very important ODE appears in all three cases.

$$X'' + \lambda^2 X = 0, \qquad 0 < x < L$$

• Homogenous BCs at $x = 0$ and $x = L$ are

$$i)\ X(0) = 0 \quad \text{or} \quad ii)\ X'(0) = 0 \quad \text{or} \quad iii)\ -kX'(0) = hX(0)$$

where h and k are positive thermophysical parameters. These are homogeneous BCs of the first kind (Dirichlet), of the second kind (Neumann), and of the third kind (Robin).
Similarly

$$i)\ X(L) = 0 \quad \text{or} \quad ii)\ X'(L) = 0 \quad \text{or} \quad iii)\ -kX'(L) = hX(L)$$

• There are *nine* combinations of these homogeneous BCs.

• Solution of ODE.

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x, \qquad 0 < x < L$$

and its derivative is

$$X'(x) = -C_1\,\lambda \sin \lambda x + C_2\,\lambda \cos \lambda x$$

• Eigenfunctions are

$$\cos \lambda x \qquad \text{and} \qquad \sin \lambda x$$

• Three combinations of the homogeneous BCs.

$$(a) \quad X'(0) = 0 \quad \text{and} \quad X(L) = 0$$

$$(b) \quad X'(0) = 0 \quad \text{and} \quad X'(L) = 0$$

$$(a) \quad X'(0) = 0 \quad \text{and} \quad -kX'(L) = hX(L)$$

• Case (a)

$$X'(0) = -C_1\lambda \sin(\lambda \times 0) + C_2\lambda \cos(\lambda \times 0) = 0$$

requires that

$$C_2\,\lambda = 0$$

Therefore $C_2 = 0$ or $\lambda = 0$. Since $\lambda = 0$ gives a trivial solution, it will be rejected, and we take $C_2 = 0$.

• Solution is

$$X(x) = C_1 \cos(\lambda x) \qquad \text{and} \qquad X'(x) = -C_1\lambda \sin(\lambda x)$$

• Apply second homogeneous Dirichlet condition to get

$$C_1 \cos(\lambda\, L) = 0$$

Either $C_1 = 0$ which gives a trivial solution, and therefore this option is rejected, or $\cos(\lambda\, L) = 0$. This is satisfied when

$$\lambda_n L = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots$$

• Eigenvalues are

$$\lambda_n = (2n-1)\frac{\pi}{2L}, \quad n = 1, 2, 3, \ldots$$

• Many math texts call λ_n^2 the eigenvalues. Most engineering texts call λ_n the eigenvalues.

• Eigenfunctions which satisfy the ODE and the two homogeneous Dirichlet conditions at $x = 0$ and $x = L$ are

$$X_n(x) = D_n \cos\left((2n-1)\frac{\pi x}{2L}\right), \quad n = 1, 2, 3. \ldots$$

where D_n are arbitrary constants.

• Case (b)

$$X'(0) \qquad \text{and} \qquad X'(L) = 0$$

The first homogeneous Neumann condition requires $C_2 = 0$ and $X(x) = C_1 \cos(\lambda x)$ and $X'(x) = -C_1 \lambda \sin(\lambda x)$ as in Case (a). The homogeneous Neumann condition at $x = L$ requires that

$$X'(L) = -C_1 \lambda \sin(\lambda L) = 0$$

Both options $C_1 = 0$ and $\lambda = 0$ give trivial solutions, therefore they are rejected.

• Eigenfunctions for Case (b) are

$$\sin(\lambda_n L) = 0, \quad n = 1, 2, 3, \ldots$$

which require

$$\lambda_n L = n\,\pi, \quad n = 1, 2, 3, \ldots$$

• Eigenvalues for Case (b) are

$$\lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

• Eigenfunctions which satisfy the ODE and the two homogeneous Neumann conditions at $x = 0$ and $x = L$ are

$$X_n(x) = D_n \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3. \ldots$$

where D_n are arbitrary constants.

• Case (c)

$$X'(0) \qquad \text{and} \qquad -\,k\,X'(L) = hX(L)$$

The first homogeneous Neumann condition requires $C_2 = 0$ and $X(x) = C_1 \cos(\lambda x)$ and $X'(x) = -C_1\lambda \sin(\lambda x)$ as in Case (a) and Case(b). The homogeneous Robin condition at $x = L$ requires that

$$-kX'(L) = hX(L) \quad \text{or} \quad -k\left[-C_1\lambda\sin(\lambda L)\right] = hC_1\cos(\lambda L)$$

The option $C_1 = 0$ gives a trivial solution, therefore it is rejected. Cancel the common C_1 and rewrite the relation as

$$\frac{h}{k}\cos(\lambda L) = \lambda\sin(\lambda L)$$

The relation is dimensional because both λ and h/k have the same units $[m^{-1}]$. The argument of the cosine and sine functions is λL which must be dimensionless.

• Characteristic Equation
Multiple through by L and define dimensionless parameters:

$$\delta = \lambda L \qquad \text{and} \qquad Bi = \frac{hL}{k} > 0$$

Now the characteristic equation becomes:

$$\delta\sin\delta = Bi\,\cos\delta \qquad \text{and} \qquad 0 < Bi < \infty$$

This is a transcendental equation. Numerical methods are required to obtain its roots δ for a given value of the parameter Bi. The Newton-Raphson iterative method can be used to obtain the roots. For a given value of Bi the infinite set of roots called eigenvalues are:

$$\delta_1 < \delta_2 < \delta_3 < \cdots < \delta_n < \delta_{n+1} < \delta_{n+2} < \cdots <$$

Lecture 2

• Makeup Lecture Number 2.

• Discuss Project Number 1. Physical interpretation of equations and the solution.

• Some Problems from Spiegel's Text:

p. 584,
A Excercises: 3

B Excercises: 2, 3
****

p. 592,
A Excercises: 1, 3
****

P. 593,
B Excercises: 2, 3
****

p. 615,
A Excercises: 1, 2, 3, 4, 5
****

p. 616-617,
B Excercises: 1, 2, 6, 7
****

p. 617,
C Excercises: 8
****

• Limiting values of parameter Bi.

• For $Bi = 0$, the characteristic equation becomes $\delta \sin \delta = 0$. Since δ cannot be set to zero (it gives a trivial solution), we set $\sin \delta = 0$. The roots (eigenvalues) are

$$\delta_n = n\,\pi, \quad n = 1, 2, 3, \ldots$$

• For $Bi = \infty$ write the equation as

$$\cos \delta = \frac{\delta}{Bi} \sin \delta$$

and now set $Bi = \infty$ to give $\cos \delta = 0$. The roots (eigenvalues) are

$$\delta_n = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots$$

• Location of Roots (Eigenvalues).

Location of the roots are easily found graphically. They lie in the intervals:

$$\begin{array}{ccc} 0 & < \; Bi < & \infty \\ 0 & < \; \delta_1 < & \pi/2 \\ \pi & < \; \delta_2 < & 3\pi/2 \\ 2\pi & < \; \delta_3 < & 5\pi/2 \\ (n-1)\pi & < \; \delta_n < & (2n-1)\pi/2 \end{array}$$

• Maple, Mathcad and Matlab can find these roots quickly and accurately.

• Difference between two consecutive eigenvalues.

$$\delta_{n+1} - \delta_n \to \pi \qquad \text{as} \qquad n \to \infty$$

• Eigenfunctions which satisfy the ODE and the homogeneous Neumann and Robin conditions at $x = 0$ and $x = L$ respectively are

$$X_n(x) = D_n \cos\left(\frac{\delta_n x}{L}\right), \quad n = 1, 2, 3. \dots$$

where D_n are arbitrary constants.

• Approximation of the First Eigenvalue, δ_1.

$$\text{As} \qquad Bi \to 0, \qquad \delta_1 \to \sqrt{Bi}$$

and

$$\text{As} \qquad Bi \to \infty, \qquad \delta_1 \to \frac{\pi}{2}$$

These limits are used to develop the following approximation proposed by M.M. Yovanovich:

$$\delta_1 = \frac{\pi/2}{\left[1 + \left(\dfrac{\pi/2}{\sqrt{Bi}}\right)^m\right]^{1/m}} \qquad \text{and} \qquad m = 2.15$$

This correlation equation is valid of all values of Bi, i.e., $0 < Bi < \infty$, and it provides acceptable accuracy for the calculation of δ_1 for most engineering applications.

• Newton-Raphson Iterative Method.

The the nth root δ_n of the arbtrary function $f(\delta)$ is obtained by means of the relation:

$$\delta_n^{(k+1)} = \delta_n^{(k)} - \frac{f(\delta_n^{(k)})}{f'(\delta_n^{(k)})}, \quad k = 1, 2, 3, \dots$$

where (k) represents the kth iteration.
For the characteristic equation, $\delta \sin \delta - Bi \cos \delta = 0$, we have the relation:

$$\delta_n^{(k+1)} = \delta_n^{(k)} - \frac{\delta_n^{(k)} \sin \delta_n^{(k)} - Bi \cos \delta_n^{(k)}}{\delta_n^{(k)} \cos \delta_n^{(k)} + (1 + Bi) \sin \delta_n^{(k)}}, \quad k = 1, 2, 3 \dots$$

The first guess of the first root $\delta_1^{(1)}$ this characteristic equation can be based on the approximation given above. Generally three to four iterations will provide very accurate values for the first root. The first guess for the second root should be based on

$$\delta_2^{(1)} = \delta_1(\text{converged value}) + \pi$$

This process can be followed to calculate all required roots which may be as many as several hundred for a particular problem. Maple and Mathcad can calculate these roots very quickly and accurately. See ME 303 Web site: Maple WS called CECART.MWS.

Lecture 4

- See Spiegel's Text: Sections 2.1 and 2.2 on pages 585-591.

- Separation of Variables Method (SVM) applied to 1D wave equation.

$$u_{xx} = \frac{1}{c^2} u_{tt}, \qquad t > 0, \qquad 0 < x < L$$

and the system constant is $c^2 = T/\rho$ where T is the tension in the elastic string and ρ is the linear density of the string. The units of $c^2\,[m^2/s^2]$ and $c\,[m/s]$. See the development of PDE in the text.

- Boundary and Initial Conditions.

The ends are fixed, therefore the homogeneous boundary conditions of the first kind (Dirichlet) are:

$$t > 0, \qquad u(0,t) = 0, \qquad u(L,t) = 0$$

The two initial conditions are based on the initial displacement and the initial velocity:

$$t = 0, \qquad 0 \le x \le L, \qquad u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x)$$

- Separated ODES.

Let $u(x,t) = X(x)T(t)$; substitute into the PDE to get the separated relationship:

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

The identity must hold for all time $t > 0$ and any value of x in interval $[0, L]$. There are three options: $lhs = rhs = (i)\; 0, (ii)\; -\lambda^2, (iii)\; \lambda^2$.

• Option (i) gives the separated ODEs:

$$X'' = 0, \qquad T'' = 0$$

and their solutions are:

$$X(x) = C_1 x + C_2, \qquad \text{and} \qquad T(t) = C_3 t + C_4$$

Both solutions are linear in time and space.

• Option (ii) gives the separated ODEs:

$$X'' + \lambda^2 X \qquad \text{and} \qquad T'' + c^2\lambda^2 T = 0$$

and their solutions are :

$$X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x) \qquad \text{and} \qquad T(t) = C_3 \cos(\lambda ct) + C_4 \sin(\lambda ct)$$

Both solutions are periodic functions of space and time respectively.

• Option (iii) gives the separated ODEs:

$$X'' - \lambda^2 X, \qquad \text{and} \qquad T'' - c^2\lambda^2 T = 0$$

and their solutions are :

$$X(x) = C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) \qquad \text{and} \qquad T(t) = C_3 \cosh(\lambda ct) + C_4 \sinh(\lambda ct)$$

Both solutions are non-periodic functions of space and time respectively.
The solutions for option (iii) can also be written in terms of exponentials:

$$X(x) = C_1 \exp(\lambda x) + C_2 \exp(-\lambda x) \qquad \text{and} \qquad T(t) = C_3 \exp(\lambda ct) + C_4 \exp(-\lambda ct)$$

• The fixed end conditions require:

$$u(0,t) = X(0)T(t) = 0 \Longrightarrow X(0) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0 \Longrightarrow X(L) = 0$$

These homogeneous BCs require $C_1 = 0$ and $C_2 = 0$ in options (i) and (iii). These solutions are not applicable to the fixed ends string.
These conditions when applied to option (ii) solution require:

$$C_1 = 0 \qquad \text{and} \qquad C_2 \sin(\lambda L) = 0$$

Since $C_2 = 0$ leads to a trivial solution, it is rejected. The other option is $\sin(\lambda L) = 0$ which leads to the eigenvalues:

$$\lambda_n = \frac{n\pi}{L}, \qquad n = 1, 2, 3, \ldots$$

Continuing with the solution. Put $C_2 C_3 = E$ and $C_2 C_4 = F$.

• General Solution of PDE.
Use Superposition Principle to write the general solution as:

$$u(x,t) = \sum_{n=1}^{\infty} \left[E_n \cos\left(\frac{n\pi ct}{L}\right) + F_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right), \quad t > 0, \quad 0 < x < L$$

Note that x/L and ct/L are dimensionless position and time.

• Fourier Coefficients
The Fourier coefficients E_n and F_n are determined by the initial displacement and velocity. The velocity for $t > 0$ and $0 \le x \le L$ is:

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} \left[E_n \left(\frac{-n\pi c}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) + F_n \left(\frac{n\pi c}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

Application of the initial conditions gives the first Fourier sine series:

$$\sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right) = f(x), \qquad 0 \le x \le L$$

Application of the initial conditions gives the second Fourier sine series:

$$\sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) F_n \sin\left(\frac{n\pi x}{L}\right) = g(x), \qquad 0 \le x \le L$$

Application of the orthogonality property of sines we find the Fourier coefficients from the integrals:

$$E_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

and

$$\left(\frac{n\pi c}{L}\right) F_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

• Illustrative Example: Plucked String at Midpoint. See page 589 of Spiegel's Text.

• Initial displacement of string. $u(x,0) = f(x)$ where

$$f(x) = \frac{2hx}{L} \quad \text{for} \quad 0 \leq x \leq \frac{L}{2}, \quad \text{and} \quad f(x) = \frac{2h(L-x)}{L} \quad \text{for} \quad \frac{L}{2} \leq x \leq L$$

• Initial velocity of displaced string. $u_t(x,0) = g(x)$ where

$$g(x) = 0$$

Since $g(x) = 0$ all Fourier coefficients $F_n = 0, n = 1, 2, 3, \ldots$.
The Fourier coefficients E_n are obtained from

$$E_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx$$

Therefore,

$$E_n = \frac{2}{L}\int_0^{L/2} \frac{2hx}{L} \sin\frac{n\pi x}{L} dx + \frac{2}{L}\int_{L/2}^{L} \frac{2h(L-x)}{L} \sin\frac{n\pi x}{L} dx$$

After some integrations we get the Fourier coeffcients for the initial displacement of the string:

$$E_n = \frac{8h}{n^2\pi^2} \sin\frac{n\pi}{2} \quad \text{for} \quad n = 1, 3, 5, \ldots \quad \text{and} \quad E_n = 0 \quad \text{for} \quad n = 2, 4, 6, \ldots$$

• Motion of the string. Solution of the undamped wave equation.

$$u(x,t) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^2} \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}$$

The first few terms of the solution are

$$\frac{u(x,t)}{h} = \frac{8}{\pi^2}\left[\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L} + \frac{1}{5^2}\sin\frac{5\pi x}{L}\cos\frac{5\pi ct}{L} - \cdots\right]$$

Observe the dimensionless displacement is $\phi = u(x,t)/h$, the dimensionless position is $\xi = x/L$ and the dimensionless time is $\tau = ct/L$ appear in the formal solution.