Week 7

Lecture 1

- Return Project 1
- Return Midterm Exam
- Exam and its solution are posted on Web site

- Examination Statistics

Table 1: Midterm Exam Summary

	Q1	Q2	Q3	Exam
Max.	30	40	30	98
Min.	8	8	10	46
Avg.	24.5	29.3	21.9	75.7
Std. Dev.	6.6	8.2	3.6	12.8

Questions were marked by: Question 1 (Yuping), Question 2 (MMY), Question 3 (Rabih)

- Read Chapter 13 of Spiegel's Text. Boundary Value Problems using Fourier Series. We will spend four to five lectures on the topics covered in this chapter.

Lecture 2

- Section 1.1: 1D Diffusion equation (Heat equation) with homogeneous Dirchlet BCs. Review of Fourier series expansions.

Lecture 3

- Discuss in detail the orthogonality of sine functions for the problem in Section 1.1 of the Spiegel text.

The initial condition leads to the Fourier sine series:

$$U(x,0) = U_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L$$

To obtain the relationship for the Fourier coefficients b_n, we must use the orthogonality property of the sine functions. Multiple the left hand side (lhs) and all terms on the right hand side (rhs) by $\sin \dfrac{m\pi x}{L}\,dx$ where $m = 1, 2, 3 \ldots$, and integrate from $x = 0$ to $x = L$. This give the relation:

$$\int_0^L U_0 \sin \frac{m\pi x}{L}\,dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\,dx$$

which is expressed in expanded form as

$$\int_0^L U_0 \sin \frac{m\pi x}{L}\,dx =$$

$$b_1 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{1\pi x}{L}\,dx \quad + \quad b_2 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{2\pi x}{L}\,dx \quad + \quad b_3 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{3\pi x}{L}\,dx \quad +$$
$$\sum_{n=4}^{\infty} b_n \int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\,dx,$$

Now systematically set $m = 1, 2, 3, \ldots$ to obtain the relations for the Fourier coefficients. For $m = 1$, we obtain the first coefficient:

$$b_1 = U_0 \frac{\displaystyle\int_0^L \sin \frac{1\pi x}{L}\,dx}{\displaystyle\int_0^L \sin^2 \frac{1\pi x}{L}\,dx}, \quad m = n = 1$$

and the remaining coefficients are equal to zero whenever $m > 1$ because of the orthogonality property of the sines. Similarly setting $m = 2$, we get the second coefficient:

$$b_2 = U_0 \frac{\displaystyle\int_0^L \sin \frac{2\pi x}{L}\,dx}{\displaystyle\int_0^L \sin^2 \frac{2\pi x}{L}\,dx}, \quad m = n = 2$$

All other terms are zero whenever $m = 1$ and $m > 2$. In general, whenever $m = n$, we obtain the relation:

$$b_n = U_0 \frac{\displaystyle\int_0^L \sin \frac{n\pi x}{L}\,dx}{\displaystyle\int_0^L \sin^2 \frac{n\pi x}{L}\,dx} = \frac{2U_0(1 - \cos n\pi)}{n\pi}, \quad n = 1, 2, 3, \ldots$$

Recall that $\cos n\pi = (-1)^n$, for $n = 1, 2, 3, \ldots$, and therefore the Fourier coefficients are obtained from

$$b_n = \frac{2U_0(1-(-1)^n)}{n\pi}, \quad n = 1, 2, 3, \ldots$$

We find that

$$b_1 = \frac{4U_0}{1\,\pi}, \quad b_3 = \frac{4U_0}{5\,\pi}, \quad b_5 = \frac{4U_0}{5\,\pi} \quad \text{for} \quad n = 1, 3, 5, \ldots, \text{odd integers}$$

and

$$b_n = 0 \quad \text{for} \quad n = 2, 4, 6, \ldots, \text{even integers}$$

The solution of the heat equation in Section 1.1 can be written as

$$\frac{U(x,t)}{U_0} = \frac{4}{\pi}\left[\frac{1}{1}\,e^{-1\,\pi^2\kappa t/L^2}\sin\frac{1\,\pi x}{L} + \frac{1}{3}\,e^{-9\,\pi^2\kappa t/L^2}\sin\frac{3\,\pi x}{L} + \frac{1}{5}\,e^{-25\,\pi^2\kappa t/L^2}\sin\frac{5\,\pi x}{L} + \ldots\right]$$

The general term of the solution can be written as

$$\frac{4}{\pi}\frac{1}{((2n-1)}\,e^{-(2n-1)^2\,\pi^2\kappa t/L^2}\sin\frac{(2n-1)\,\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

The nondimensional dependent and independent parameters are: $U(x,t)/U_0$, $\xi = x/L$ and $\tau = \kappa t/L^2$.

Discussed the convergence of solution for early times. Overshoot (Gibbs phenemenon) occurs at the end points $x = 0$ and $x = L$. As more terms of the summation are used, the overshoot decreases and moves to the end points. This can be shown by means of Maple and Mathcad.
