Laplace Transform Method applied to the one-dimensional diffusion equation in half-space. Let $u = u(x,t)$ be the temperature which is the solution of the diffusion equation:

$$u_{xx} = \frac{1}{\alpha} u_t, \quad t > 0, \quad x > 0$$

The initial condition for $x \geq 0$ is $u(x,0) = 0$. The boundary condition for all $t > 0$ as $x \to \infty$ is $u(x,t) \to 0$. The boundary condition at the surface $x = 0$ can be one of the following three conditions:

i) Dirichlet condition:

$$u(0,t) = u_0$$

ii) Neumann condition:

$$u_x(0,t) = -\frac{q_0}{k}$$

where q_0 is the constant incident heat flux and k is the constant thermal conductivity.

iii) Robin condition:

$$u_x(0,t) = -\frac{h}{k}\left[u_f - u(0,t)\right]$$

where h is the constant heat transfer coefficient and u_f is the constant temperature of the fluif which heat the surface.

The Laplace transform of the given PDE gives the ODE

$$\frac{d^2U}{dx^2} = \frac{1}{\alpha}\left[sU - u(x,0)\right]$$

where $U(x,s)$ is the Laplace transform of the temperature:

$$U(x,s) = \mathcal{L}\left\{u(x,t)\right\}$$

Appyling the IC: $u(x,0) = 0$ gives the ODE:

$$\frac{d^2U}{dx^2} = \frac{s}{\alpha} U = 0, \quad x > 0$$

whose solution is

$$U = C_1 e^{\sqrt{\frac{s}{\alpha}}x} + C_2 e^{-\sqrt{\frac{s}{\alpha}}x}$$

The boundary condition at $x = \infty$ requires that $u(\infty) = 0$ or that $U(\infty, s) = 0$ which requires that the constant $C_1 = 0$. The solution and its derivative are

$$U = C_2 e^{-\sqrt{\frac{s}{\alpha}}x} \quad \text{and} \quad \frac{dU}{dx} = -C_2\sqrt{\frac{s}{\alpha}} e^{-\sqrt{\frac{s}{\alpha}}x}$$

The second constant of integration can be obtained by the application of one of the three boundary conditions at $x = 0$. Now, we find the Laplace transform of the three boundary conditions.

i) Dirichlet condition

$$\mathcal{L}\{u(0,t)\} = \mathcal{L}\{u_0\} = \frac{u_0}{s}, \quad s > 0$$

ii) Neumann condition

$$\mathcal{L}\{u_x(0,t)\} = \mathcal{L}\left\{\frac{q_0}{k}\right\}$$

which requires

$$\frac{dU(0,s)}{dx} = -\frac{q_0}{ks}, \quad s > 0$$

iii) Robin condition

$$\mathcal{L}\{u_x(0,t)\} = \mathcal{L}\left\{\frac{h}{k}\left[u_f - u(0,t)\right]\right\}$$

which requires

$$\frac{dU(0,s)}{dx} = -\frac{h}{k}\frac{u_f}{s} + \frac{h}{k}U(0,s), \quad s > 0$$

Solutions in the transform domain for the three boundary conditions. Here the three boundary conditions at $x = 0$ are applied to find the constant C_2 and the three solutions.

i) Dirichlet boundary condition

$$C_2 \, e^{-\sqrt{\frac{s}{\alpha}}x}\Big|_{x=0} = C_2 = \frac{u_0}{s}$$

The solution is

$$U(x,s) = \frac{u_0}{s} e^{-\sqrt{\frac{s}{\alpha}}x}, \quad s > 0, \quad x \geq 0$$

ii) Neumann boundary condition

$$-C_2\sqrt{\frac{s}{\alpha}}\, e^{-\sqrt{\frac{s}{\alpha}}x}\Big|_{x=0} = -C_2\sqrt{\frac{s}{\alpha}} = -\frac{q_0}{ks}$$

Solving for C_2 gives

$$C_2 = \frac{q_0}{k}\frac{\sqrt{\alpha}}{s^{3/2}}$$

The solution is

$$U(x,s) = \frac{q_0}{k}\frac{\sqrt{\alpha}}{s^{3/2}}\, e^{-\sqrt{\frac{s}{\alpha}}x}, \quad s>0, \quad x \geq 0$$

iii) Robin boundary condition

$$-C_2\sqrt{\frac{s}{\alpha}}\, e^{-\sqrt{\frac{s}{\alpha}}x}\Big|_{x=0} = -\frac{h}{k}\frac{u_f}{s} + \frac{h}{k}\, U(x,s)|_{x=0}$$

Solving for C_2 gives

$$C_2 = \frac{\dfrac{hu_f}{ks}}{\dfrac{h}{k} + \sqrt{\dfrac{s}{\alpha}}}$$

The solution is

$$U(x,s) = \frac{\dfrac{hu_f}{ks}}{\dfrac{h}{k} + \sqrt{\dfrac{s}{\alpha}}} e^{-\sqrt{\frac{s}{\alpha}}x}, \quad s>0, \quad x \geq 0$$

Solutions for the three boundary conditions. Laplace transform tables can be used to get the solutions $u(x,t)$.