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Summary of Moving Source Solutions

The governing partial differential equation based on a heat balance on a fixed cartesian differential control volume is

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \pm \frac{V}{\alpha} \frac{\partial \theta}{\partial x} = 0 \tag{1}$$

where $\theta = T(x, y, z) - T_\infty$, the temperature excess relative to the undisturbed temperature T_∞; $\alpha = k/(\rho\ c_p)$ is the thermal diffusivity of the material; and the heat source is moving with a constant velocity V along the $x-$axis in the positive or negative direction respectively.

The transformation:

$$\theta(x, y, z) = \exp\left(-\frac{V}{2\ \alpha}\ x\right)\ \phi(x, y, z) \tag{2}$$

leads to the following Helmholtz equation for the dependent parameter ϕ:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \left(\frac{V}{2\ \alpha}\right)^2 \phi = 0 \tag{3}$$

for the case where the heat source is moving along the positive $x-$axis.

Three special cases arise from the above general equation: i) plane source, ii) line source and iii) point source.

Plane Source

The plane source solution $\theta(x)$ is obtained from the solution of

$$\frac{d^2 \phi}{dx^2} - \left(\frac{V}{2\ \alpha}\right)^2 \phi = 0 \tag{4}$$

which is

$$\phi(x) = C_1\ \exp\left(-\frac{V}{2\alpha}\ x\right) + C_2\ \exp\left(\frac{V}{2\alpha}\ x\right) \tag{5}$$

Thus the solution for the plane source is

$$\theta(x) = C_1\ \exp\left(-\frac{V}{\alpha}\ x\right) + C_2 \tag{6}$$

Line Source

The moving line source produces a temperature field which is independent of z but it is a function of $r = \sqrt{x^2 + y^2}$; therefore the Helmholtz equation becomes:

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\,\frac{d\phi}{dr} - \left(\frac{V}{2\,\alpha}\right)^2 \phi = 0 \tag{7}$$

which is the modified Bessel equation of order zero. The solution is

$$\phi = C_1\, I_0\left(\frac{V}{2\alpha}\, r\right) + C_2\, K_0\left(\frac{V}{2\alpha}\, r\right) \tag{8}$$

where $I_0\left(\frac{V}{2\alpha}\, r\right)$ and $K_0\left(\frac{V}{2\alpha}\, r\right)$ are the modified Bessel functions of the first and second kind of zero order respectively.

Since $I_0\left(\frac{V}{2\alpha}\, r\right) \to \infty$ as $r \to \infty$ and the solution must be bounded, we must reject it as part of the solution, thus the required solution for the line source is

$$\theta(r) = C_2\ \exp\left(-\frac{V}{2\alpha}\, x\right)\ K_0\left(\frac{V}{2\alpha}\, r\right) \tag{9}$$

Point Source

If a point source moves through a stationary medium, the temperature field will depend on $r = \sqrt{x^2 + y^2 + z^2}$, and the Helmholtz equation becomes

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\,\frac{d\phi}{dr} - \left(\frac{V}{2\,\alpha}\right)^2 \phi = 0 \tag{10}$$

which has the solution:

$$\phi(r) = C_1\,\frac{1}{r}\ \exp\left(-\frac{V}{2\alpha}\, r\right) + C_2\,\frac{1}{r}\ \exp\left(\frac{V}{2\alpha}\, r\right) \tag{11}$$

To insure that the solution will be bounded at remote points, ie, as $r \to \infty$ we must reject the second part of the above solution by setting $C_2 = 0$.

The solution for the moving point source is therefore

$$\theta(r) = C_1\,\frac{1}{r}\ \exp\left(-\frac{V}{2\alpha}\,(r + x)\right) \tag{12}$$