UNIVERSITY OF WATERLOO

Department of Mechanical Engineering

ME 303 Advanced Engineering Mathematics

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Nonhomogeneous PDE

Given the transient conduction equation with distributed volumetric heat sources $S > 0$:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{S}{k} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}, \quad t > 0, \quad 0 < x < L$$

with $\theta(x,t) = T(x,t) - T_\infty$, where T_∞ represents the ambient temperature.

Boundary Conditions (BCs) and Initial Condition (IC)

The BCs and IC are:

$$x = 0, \quad \frac{\partial\theta(0,t)}{\partial x} = 0, \quad \text{and} \quad x = L, \quad \theta(L,t) = \theta_L = T_L - T_\infty$$

and

$$\theta(x,0) = 0 \quad \text{for} \quad 0 \leq x \leq L$$

The BCs are homogeneous Neumann and nonhomogeneous Dirichlet respectively, and the IC is homogeneous.

Solution Procedure

The solution will be written as

$$\theta(x,t) = v(x) + w(x,t)$$

where $v(x)$ is the steady-state solution for $t \to \infty$, and $w(x,t)$ is the auxiliary transient solution. Substitute into the given PDE to get

$$v_{xx} + w_{xx} + \frac{S}{k} = \frac{1}{\alpha}w_t$$

Now separate the above equation into two equations which are:
1) Nonhomogeneous ODE:

$$v_{xx} = -\frac{S}{k}, \quad 0 < x < L$$

2) Homogeneous PDE:

$$w_{xx} = \frac{1}{\alpha} w_t, \quad t > 0, \quad 0 < x < L$$

Solution of Nonhomogeneous ODE

The solution of the nonhomogeneous ODE is

$$\frac{dv}{dx} = -\frac{Sx}{k} + C_1 \quad \text{and} \quad v(x) = -\frac{Sx^2}{2k} + C_1 x + C_2$$

The boundary conditions are

$$x = 0, \quad \frac{\partial \theta}{\partial x} = \frac{dv}{dx} + \frac{\partial w}{\partial x} = 0$$

Therefore both

$$\frac{dv}{dx} = 0 \quad \text{and} \quad \frac{\partial w}{\partial x} = 0$$

At the other boundary

$$x = L, \quad \theta(L, t) = v(L) + w(L, t) = \theta_L$$

We must select

$$v(L) = \theta_L \quad \text{and} \quad w(L, t) = 0$$

The two boundary conditions applied to the solution $v(x)$ requires that

$$C_1 = 0 \quad \text{and} \quad C_2 = \frac{SL^2}{2k} + \theta_L$$

The steady-state solution is therefore

$$v(x) = -\frac{Sx^2}{2k} + \frac{SL^2}{2k} + \theta_L, \quad 0 \le x \le L$$

Solution of Homogeneous PDE

The transient problem is

$$w_{xx} = \frac{1}{\alpha} w_t, \quad t > 0, \quad 0 < x < L$$

with homogeneous boundary conditions:

$$x = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{and} \quad x = L, \quad w(L, t) = 0$$

and nonhomogeneous initial condition:

$$t = 0, \quad w(x, 0) = \theta(x, 0) - v(x) = -v(x), \quad 0 \leq x \leq L$$

The Separation of Variables Method (SVM) leads to the solution

$$w(x, t) = T(t)X(x) = e^{-\lambda^2 \alpha t} \left[C \cos \lambda x + D \sin \lambda x\right]$$

and

$$\frac{\partial w}{\partial x} = T(t)X'(x) = e^{-\lambda^2 \alpha t} \left[-C\lambda \sin \lambda x + D\lambda \cos \lambda x\right]$$

The two homogeneous boundary conditions require

$$x = 0, \quad X'(0) = 0 \quad \text{and} \quad x = L, \quad X(L) = 0$$

The first boundary condition (homogeneous Neumann condition) requires $D = 0$, which removes the sine function from the solution. The second boundary condition (homogeneous Dirichlet condition) requires

$$C \cos \lambda L = 0, \quad \text{thus} \quad \cos \lambda L = 0, \quad \text{therefore} \quad \lambda_n = \frac{n\pi}{2L}, \quad 1, 3, 5, \ldots, \text{odd integers}$$

or

$$\lambda_n = \frac{(2n-1)\pi}{2L}, \quad 1, 2, 3, \ldots$$

for all positive integers.

The general term of the auxiliary transient solution is

$$w_n(x, t) = C_n e^{-(2n-1)^2 \pi^2 \alpha t / (4L^2)} \cos\left(\frac{(2n-1)\pi x}{2L}\right), \quad n = 1, 2, 3 \ldots$$

Application of the superposition principle gives the general solution:

$$w(x, t) = \sum_{n=1}^{\infty} C_n e^{-(2n-1)^2 \pi^2 \alpha t / (4L^2)} \cos\left(\frac{(2n-1)\pi x}{2L}\right), \quad t > 0, \quad 0 < x < L$$

Fourier Coefficients for Temperature

To find the Fourier coefficients C_n we apply the initial condition to get the Fourier cosine series:

$$-v(x) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{(2n-1)\pi x}{2L}\right), \quad t > 0, \quad 0 < x < L$$

Application of the orthogonality property of cosines gives the relation:

$$C_n = \frac{2}{L}\int_0^L -v(x)\cos\left(\frac{(2n-1)\pi x}{2L}\right) dx, \quad n = 1, 2, 3 \ldots$$

Substitution of the steady-state solution and integrating we get, after some calculus, the final relationship for the Fourier coefficients:

$$C_n = \frac{(-1)^n 16}{(2n-1)^3 \pi}\left\{\frac{SL^2}{\pi^2 k} + (T_L - T_\infty)\left(n^2 - n + 1/4\right)\right], \quad n = 1, 2, 3, \ldots$$

- Alternative form of the Fourier coefficients for temperature:

$$C_n = -2\left[\frac{S}{kL\lambda_n^3} + \frac{\theta_L}{L\lambda_n}\right]\sin\lambda_n L$$

and

$$\lambda_n L = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots$$

Note that the units of the Fourier coefficients are K.

The $\theta(x, t)$ solution can be written in the semi-dimensionless form:

$$\theta(x,t) = \frac{SL^2}{2k}(1-\xi^2) + \sum_{n=1}^{\infty} C_n e^{-(2n-1)^2(\pi^2/4)\tau} \cos\left(\frac{(2n-1)\pi}{2}\xi\right), \quad \tau > 0, \quad 0 < \xi < 1$$

with dimensionless time $\tau = \alpha t/L^2$ and dimensionless position $\xi = x/L$.

The temperature $T(\xi, \tau)$ is obtained from

$$T(\xi,\tau) = \theta(\xi,\tau) + T_\infty = v(\xi) + w(\xi,\tau) + T_\infty$$

Instantaneous Heat Transfer Rate

The instantaneous heat transfer rate through the boundary at $x = L$ can be obtained from the Fourier Law of Conduction:

$$Q(t) = -kA\left(\frac{\partial\theta(x,t)}{\partial x}\right)_{x=L} = -\frac{kA}{L}\left(\frac{\partial\theta(\xi,\tau)}{\partial\xi}\right)_{\xi=1}$$

which gives

$$Q(t) = SAL + \frac{kA}{L}\sum_{n=1}^{\infty} E_n \, e^{-(2n-1)^2(\pi^2/4)\tau}, \quad \tau > 0$$

where the conduction area is $A = \pi D^2/4$, and the Fourier coefficients E_n for the heat flow rate are obtained from the relation:

$$E_n = (-1)^{n+1} \, C_n \frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3 \ldots$$

The final form of the expression for the Fourier coefficients for the heat flow rate is

$$E_n = \frac{-8}{(2n-1)^2}\left[\frac{SL^2}{\pi^2 k} + (T_L - T_\infty)(n^2 - n + 1/4)\right], \quad n = 1, 2, 3 \ldots$$

- Alternative form of Fourier coefficients for $Q(t)$

$$F_n = -C_n \, \lambda_n \sin \lambda_n L = 2\lambda_n \left[\frac{S}{kL\lambda_n^3} + \frac{\theta_L}{L\lambda_n}\right] \sin^2 \lambda_n L$$

where $E_n = LF_n$, and

$$\lambda_n L = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots$$

Maple Solutions

The complete solution of the given nonhomogeneous PDE is presented in the Maple worksheets: **PROJ2S99SOL.MWS** and **PROJ2S99SOL2.MWS**. The temperature $\theta(x, t)$ or $\theta(\xi, \tau)$ and the instantaneous heat transfer rate through the boundary at $x = L$ or $\xi = 1$ are presented in the Maple worksheets.

System Parameter Values

$D = 5\, mm$
$L = 100\, mm$
$k = 80\, W/(m \cdot K)$
$\alpha = 1.2 \times 10^{-5}\, m^2/s$
$S = 2 \times 10^6\, W/m^3$
$T_L = 70°C$
$T_\infty = 20°C$

Calculations of Instantaneous Heat Transfer Rates

Table 1: Calculated Heat Flow Rates, $Q(\tau)$, $[W]$

τ	$Q(\tau)$
0.01	-3.988
0.1	0.0001216
1.0	3.524
∞	3.927

The calculated heat flow rates presented in Table 1 are based on the summation of the first 200 terms of the transient part of the solution. An examination of the solution reveals that the series converges very quickly for larger values of the dimensionless time τ, and therefore, only a few terms are required to provide accurate values when $\tau > 0.1$.

The heat flow rate is *into* the rod when $\tau = 0.01$. At $\tau = 0.1$, the heat flow rate is *out* of the rod and its value is small. When $\tau = 1.0$, the heat flow rate is approximately 90% of the steady-state value of $Q(\infty) = 3.927$.

Plots of Temperature

The plots of the steady-state solution $T(\xi, \infty) = v(\xi) + T_\infty$, and the transient solution $\theta(\xi, \tau) + T_\infty$ for dimensionless times of $\tau = 0.01, 0.1,$ and 1.0 are presented in the Maple worksheets:
PROJ2S99SOL.MWS and **PROJ2S99SOL2.MWS**.
