**ME 353 M.M. Yovanovich**

# General Solution of Poisson Equation for Plane Wall, Long Solid Circular Cylinder and Solid Sphere

The general Poisson equation $\nabla^2 T = -\mathcal{P}/k$ with appropriate boundary conditions, and the general solution which is valid for the plane wall, long solid circular cylinder and solid sphere is presented below. The temperature is one-dimensional, ie $T = T(u)$ where $u$ represents the $x-$ and $r-$ coordinates in the plane wall, cylinder and sphere. The distributed volumetric heat sources $\mathcal{P}$ and the thermal conductivity $k$ are constant.

**Differential Equation**

$$\frac{1}{u^n}\frac{d}{du}\left(u^n \frac{dT}{du}\right) = -\frac{\mathcal{P}}{k}, \qquad 0 \le u \le b, \qquad n = 0, 1, 2$$

**Boundary Conditions**

$$u = 0, \quad \frac{dT}{du} = 0, \qquad \text{symmetry condition}$$

$$u = b, \quad \frac{dT}{du} = -\frac{h}{k}\left[T(b) - T_f\right]$$

The parameters in the previous equations are defined below for the three geometries.

| Geometry | Plane Wall | Cylinder | Sphere |
|---|---|---|---|
| $u =$ | x | r | r |
| $b =$ | L | b | b |
| $n =$ | 0 | 1 | 2 |

### Solution

Integrating once gives the temperature gradient:

$$\frac{dT}{du} = -\frac{\mathcal{P}}{k}\frac{u}{n+1} + \frac{C_1}{u^n}$$

with the first integration constant $C_1$.
The second integration gives the temperature distribution:

$$T(u) = -\frac{\mathcal{P}}{k}\frac{u^2}{2(n+1)} + C_1\frac{u^{1-n}}{(1-n)} + C_2$$

with the second integration constant $C_2$.
To satisfy the symmetry condition at $u = 0$, the first integration constant must be set to zero, i.e., $C_1 = 0$. The second boundary condition leads to the relation:

$$-\frac{\mathcal{P}b}{k(n+1)} = -\frac{h}{k}\left[-\frac{\mathcal{P}b^2}{2k(n+1)} + C_2 - T_f\right]$$

The previous relation gives the second constant of integration:

$$C_2 = T_f + \frac{\mathcal{P}b}{h(n+1)} + \frac{\mathcal{P}b^2}{2k(n+1)}$$

### General Temperature Distribution

The general temperature distribution can be written as

$$T(u) - T_f = \frac{\mathcal{P}}{2k(n+1)}(b^2 - u^2) + \frac{\mathcal{P}b}{h(n+1)}, \qquad 0 \le u \le b$$

which is valid for $n = 0$ (plane wall), $n = 1$ (long solid cylinder), and $n = 2$ (solid sphere).

### Wall or Surface Temperature Drop

The temperature drop from the wall (or surface) to the fluid is obtained from the general solution by setting $u = b$ and $T(b) = T_s$. Therefore we get

$$T_s - T_f = \frac{\mathcal{P}b}{h(n+1)}$$

### Centerline or Axis Temperature Drop

The temperature drop from the centerline or the axis where the maximum temperature occurs to the fluid is obtained from the general solution by setting $u = 0$ and $T(0) = T_{\max}$. Therefore we get

$$T_{\max} - T_f = \frac{\mathcal{P}b^2}{2k(n+1)} + \frac{\mathcal{P}b}{h(n+1)}$$

### Solid Temperature Drop

The temperature drop across the solid is

$$T_{\max} - T_{\mathrm{s}} = \frac{\mathcal{P}b^2}{2k(n+1)}$$

### Ratio of Solid to Film Temperature Drops

The ratio of the solid temperature drop to the film temperature drop is obtained from

$$\frac{\Delta T_{\mathrm{solid}}}{\Delta T_{\mathrm{film}}} = \frac{T_{\max} - T_{\mathrm{s}}}{T_{\mathrm{s}} - T_f} = \frac{1}{2}\frac{hb}{k} = \frac{1}{2}Bi$$

where $Bi$ is the Biot number.