Week 8

Lecture 1

Project 2 will be handed out Wednesday, October 29; due date is Wednesday, November 5; transient conduction in half-space and full-space; Dirichlet, Neumann and Robin solutions; see page 239 of text and the Web site; approximations of erf(x), erfc(x), and its inverse: $y = \text{erfc}(x)$ and $x = \text{erfc}^{-1}(y)$; ranges: $0 \le y < 1$, $x \ge 0$; transient conduction from isothermal sphere into full-space; analytical solution for Dirichlet BC obtained by Laplace transform method: $\boxed{T(r,t) = (a/r)T_s \, \text{erfc}((r-a)/2\sqrt{\alpha t})}$; instantaneous heat transfer rate from sphere into full-space: $\boxed{Q^\star_{\mathcal{L}} = \iint_{A_{\text{sphere}}} -k\partial T(r=a,t)/\partial r dA}$ results in linear superposition of steady-state and transient parts: $\boxed{Q^\star_{\mathcal{L}} = Q^\star_{ss} + Q^\star_t}$; several choices for characteristic body length: $\mathcal{L} = a, D$ or $\sqrt{A}$ where A is the total surface area of the body;

$$Q^\star_{\sqrt{A}} = S^\star_{\sqrt{A}} + \frac{1}{\sqrt{\pi}\sqrt{Fo_{\sqrt{A}}}}$$

where $S^\star_{\sqrt{A}} = S\sqrt{A}/(k(T_s - T_i)A)$ is the dimensionless shape factor and the dimensionless time is called the Fourier number: $Fo = \alpha t/(\sqrt{A})^2$.

Lecture 2

Some values of the dimensionless shape factor for several body shapes and aspect ratios (AR = height − to − width) are given below:

Geometry	$S^{\star}_{\sqrt{A}}$	n
Circular Disk	3.192	1.10
Rectangular Strip	3.303	1.07
Square Disk	3.343	1.05
Cube	3.388	1.05
Cuboid ($AR = 2$)	3.406	1.03
Sphere	3.545	1
Oblate ($AR = 0.5$)	3.529	0.994
Prolate ($AR = 2$)	3.564	0.99
Cuboid ($AR = 10$)	3.945	0.96
Prolate ($AR = 10$)	4.195	0.87

the correlation parameter n appears in the correlation equation and values are given in the Table:

$$Q^{\star}_{\sqrt{A}} = \left[\left(S^{\star}_{\sqrt{A}} \right)^n + \left(\frac{1}{\sqrt{\pi}\sqrt{Fo_{\sqrt{A}}}} \right)^n \right]^{1/n}$$

parameter $n = 1$ for the sphere and has the range: $0.87 \leq n \leq 1.10$ for a range of geometries.

Internal transient conduction: plane wall $T(x,t)$, long solid circular cylinder $T(r,t)$ and solid sphere $T(r,t)$; See text and Web site for Maple worksheets for Heisler cooling charts for one-dimensional diffusion; dimensionless temperature: $\phi(\zeta, Fo)$ depends on dimensionless position: $\zeta = x/L$ for wall of thickness: $2L$; $\zeta = r/a$ for cylinder and sphere of radius a; and dimensionless time: $Fo = \alpha t/\mathcal{L}^2$ where $\mathcal{L} = L$ or a; dimensionless temperature is defined as:

$$\phi_h = \frac{T(\zeta, Fo) - T_i}{T_f - T_i} \qquad \text{for heating}$$

and

$$\phi_c = \frac{T(\zeta, Fo) - T_f}{T_i - T_f} \qquad \text{for cooling}$$

General form of solutions

$$\phi(\zeta, Fo) = \sum_{n=1}^{\infty} A_n \exp\left(-\delta_n^2 Fo\right) S(\delta_n \zeta) \qquad Fo > 0$$

where A_n are the temperature Fourier coefficients; $S(\delta_n \zeta)$ is the space-dependent function, and $\exp(-\delta_n^2 Fo)$ is the time-dependent function; the eigenvalues: δ_n are the positive, real roots of the characteristic equations:

$$\delta_n \sin \delta_n = Bi \cos \delta_n \qquad \text{plane wall}$$

and

$$\delta_n J_1(\delta_n) = Bi J_0(\delta_n) \qquad \text{cylinder}$$

where J_0 and J_1 are Bessel functions of the first kind of order 0 and 1; and

$$\delta_n \cos \delta_n = (1 - Bi) \sin \delta_n \qquad \text{sphere}$$

where the Biot number: $Bi = h\mathcal{L}/k$ where $0 < Bi < \infty$.

Space-dependent Functions

$$S(\delta_n \zeta) = \cos(\delta_n \zeta) \qquad \text{plane wall}$$

and

$$S(\delta_n \zeta) = J_0(\delta_n \zeta) \qquad \text{cylinder}$$

and

$$S(\delta_n \zeta) = \frac{\sin(\delta_n \zeta)}{(\delta_n \zeta)} \qquad \text{sphere}$$

Fourier Temperature Coefficients: A_n

$$A_n = \frac{2 \sin \delta_n}{\delta_n + \sin \delta_n \cos \delta_n} \qquad \text{plane wall}$$

and

$$A_n = \frac{2 J_1(\delta_n)}{\delta_n \left[J_0^2(\delta_n) + J_1^2(\delta_n) \right]} \qquad \text{cylinder}$$

and

$$A_n = \frac{2(\sin \delta_n - \delta_n \cos \delta_n)}{\delta_n - \sin \delta_n \cos \delta_n} \qquad \text{sphere}$$

Long Time Solutions

For $Fo > Fo_c$ where $Fo_c = 0.24, 0.21, 0.18$ for plane wall, cylinder and sphere, respectively, the general solution converges to the first term of the summation, i.e.,

$$\phi(\zeta, Fo) = A_1 \exp(-\delta_1 Fo) S(\delta_1 \zeta)$$

and the first eigenvalue can be approximated by the correlation equation:

$$\delta_1 = \delta_{1,\infty} \left[1 + \left(\frac{\delta_{1,\infty}}{\delta_{1,0}} \right)^n \right]^{-1/n}$$

where $\delta_{1,0}$ is the asymptotic value as $Bi \rightarrow 0$ and $\delta_{1,\infty}$ is the asymptotic value as $Bi \rightarrow \infty$; the values are given in the following table:

Geometry	Wall	Cylinder	Sphere
$\delta_{1,0}$	$\sqrt{Bi}$	$\sqrt{2Bi}$	$\sqrt{3Bi}$
$\delta_{1,\infty}$	$\pi/2$	2.4048255	π
n	2.139	2.238	2.314

Lecture 3

Hand out Project 2. Discuss Project 2 and show example of numerical solution on Web site.

Lumped Capacitance Model

For $Bi < 0.2$ and all time $Fo > 0$, general solution reduces to the first term of the summation where $A_1 \rightarrow 1$ and $S(\delta_1 \zeta) \rightarrow 1$; the three solutions become:

$$\phi(Fo) = \exp(-BiFo) \qquad \text{wall}$$

and

$$\phi(Fo) = \exp(-2BiFo) \qquad \text{cylinder}$$

and

$$\phi(Fo) = \exp(-3BiFo) \qquad \text{sphere}$$

Heat Loss Fraction

$$\frac{Q}{Q_i} = 1 - \sum_{n=1}^{\infty} B_n \exp\left(-\delta_n^2 Fo\right)$$

where $Q = E_i - E(t) = E(0) - E(t)$ and $E_i = \rho c_p V(T_i - T_f)$ is the initial internal energy with respect to the fluid temperature for cooling and $E(t) = \rho c_p V(\bar{T} - T_f)$ is the internal energy at some time $t > 0$. $\bar{T}$ is the average temperature of the system at some time $t > 0$.

Fourier Coefficients for Heat Loss Fraction

$$B_n = A_n \frac{\sin \delta_n}{\delta_n} \qquad \text{wall}$$

and

$$B_n = 2A_n \frac{J_1(\delta_n)}{\delta_n} = \frac{4Bi^2}{\delta_n^2(\delta_n^2 + Bi^2)} \qquad \text{cylinder}$$

and

$$B_n = \frac{6Bi^2}{\delta_n^2(\delta_n^2 + Bi^2 - Bi)} \qquad \text{sphere}$$

For $Fo > Fo_c$, summation reduces to the first term, and

$$\frac{Q}{Q_i} = 1 - B_1 \exp(-\delta_1 Fo)$$

See text and Web site for Maple worksheets for plane wall, long cylinder and sphere; See section 5.8 of text for multidimensional effects; superposition of solutions; see example 5.6 and the corresponding Maple worksheet; you are not responsible for section 5.9.
