Problem: 8-1

Show: For a perfect gas undergoing an isentropic process show that $\dfrac{T_2}{T_1} = \left(\dfrac{p_2}{p_1}\right)^{(k-1)/k}$, where $k = c_p/c_v$

Gibbs equation for simple compressible substances can be written as

$$ds = \frac{1}{T}du + \frac{P}{T}dv$$

or for an isentropic process

$$\frac{1}{T}du = -\frac{P}{T}dv \qquad (1)$$

for a perfect gas

$$\begin{aligned} du &= c_v(T)dT \\ Pv &= RT \\ v &= R\frac{T}{P} \\ dv &= R\frac{dT}{P} - RT\frac{dP}{P^2} \end{aligned}$$

substitute into Eq. 1

$$\begin{aligned} c_v(T)\frac{dT}{T} &= -R\left(\frac{dT}{T} - \frac{dP}{P}\right) \\ c_v\int_1^2 \frac{dT}{T} &= -R\int_1^2 \frac{dT}{T} + R\int_1^2 \frac{dP}{P}, \qquad \text{for } c_v = \text{const} \\ (c_v + R)\ln\frac{T_2}{T_1} &= R\ln\frac{P_2}{P_1} \\ \frac{T_2}{T_1} &= \left(\frac{P_2}{P_1}\right)^{R/(c_v+R)} \qquad (2) \end{aligned}$$

but $h \equiv u + Pv$, therefore for a perfect gas and constant specific heats

$$\begin{aligned} dh &= du + d(Pv) = du + d(RT) \\ c_p\cancel{d}T &= c_v\cancel{d}T + R\cancel{d}T \\ c_p &= c_v + R \end{aligned}$$

Therefore

$$\frac{R}{c_v + R} = \frac{c_p - c_v}{c_p} = \frac{k-1}{k}$$

and Eq. (2) becomes

$$\boxed{\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{(k-1)/k}} \qquad \Leftarrow$$